CONFORMAL OPERATORS ON WEIGHTED FORMS; THEIR DECOMPOSITION AND NULL SPACE ON EINSTEIN MANIFOLDS

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ABSTRACT. There is a class of Laplacian like conformally invariant differential operators on differential forms L_k^ℓ which may be considered the generalisation to differential forms of the conformally invariant powers of the Laplacian known as the Paneitz and GJMS operators. On conformally Einstein manifolds we give explicit formulae for these as explicit factored polynomials in second order differential operators. In the case the manifold is not Ricci flat we use this to provide a direct sum decomposition of the null space of the L_k^ℓ in terms of the null spaces of mutually commuting second order factors.

1. INTRODUCTION

On Riemannian and pseudo-Riemannian manifolds a natural differential operator is said to be *conformally invariant* if it descends to a well-defined differential operator on *conformal manifolds*, that is manifolds equipped only with an equivalence class c of metrics, where $g, g' \in c$ means that $g' = fg$ for some positive function f. This special class of operators have long held a central place in mathematics and physics. For example they govern the behaviour of massless particles, and have desirable properties on conformally compact manifolds, as for example used in general relativity [11, 42]. They play a role in curvature prescription, in extremal problems for metrics both in Riemannian geometry as well as in string and brane theories [7, 10, 12, 13, 44]. Via the Fefferman bundle and metric, conformal structure and the corresponding differential operators are also important in complex and CR geometry [21, 25].

A rich programme surrounds the conformally invariant differential operators P_{2k} with leading term a power of the Laplacian Δ^k, see e.g. [5, 8, 20, 35, 37, 39] and references therein. This family of operators on scalar functions (or more accurately conformal densities) consists of the second order *conformal Laplacian* [46], (also called the *Yamabe operator*), the 4th order Paneitz operator [43], and higher order *GJMS operators* of [34].

Many of the important features of the GJMS family are shared by a larger family of operators L_k^ℓ introduced in [6]. (Related operators of order 2, 4 and 6 were constructed in [3]. See also [16] a the role of the order 2 operator in Physics.) Each of these act on density-valued differential forms $\mathcal{E}^k[w]$ (for the notation see Section 2) and, up to a non-zero constant multiple, takes the form

$$\underbrace{(n-2k+2\ell)}_{-2u}(\delta d)^\ell + \underbrace{(n-2k-2\ell)}_{-2w}(d\delta)^\ell + \text{ lower order terms,}$$

and carries $\mathcal{E}^k[w]$ to $\mathcal{E}^k[u]$. Here d is the exterior derivative, δ its formal adjoint and this formula follows from the formulae for the operators on the sphere ([4], Remark 3.30). At the $k=0$ specialisation (and with restrictions on ℓ in the case of even dimensions) these are the usual GJMS operators of [34], see [6]. For other k and when neither u or w is zero they are evidently still *Laplacian like* in the sense that the composition $(w\delta d + ud\delta)\circ L^k_\ell$ takes the form

$$(w\delta d + ud\delta)\circ L^k_\ell \sim \Delta^{\ell+1} + \text{ lower order terms},$$

where the "$\sim$" means up to a non-zero constant multiple. This means immediately that the operators L^k_ℓ are elliptic in the case of Riemannian signature (conformal structures), or hyperbolic/ultrahyperbolic in the case of other signatures. If u or w is zero (and $k\neq 0,n$) then these operators instead arise in conformally invariant differential complexes (namely *detour complexes*) with the corresponding properties; for example the complexes concerned are elliptic on Riemannian signature backgrounds. In all cases the operators yield globally conformally invariant pairings on compactly supported sections

$$\mathcal{E}^k[\ell+k-\frac{n}{2}] \ni \varphi,\psi \mapsto \langle\varphi,\psi\rangle := \int_M \varphi\cdot L^\ell_k\psi\, d\mu_{\boldsymbol{g}}, \tag{1}$$

where $\varphi\cdot L^\ell_k\psi\in\mathcal{E}[-n]$ denotes a complete metric contraction between φ and $L^\ell_k\psi$, and $d\mu_{\boldsymbol{g}}$ is the conformal measure. The operators L^ℓ_k are formally self-adjoint, and so this pairing is symmetric.

It is shown in [6] that the family L^k_ℓ leads to operators which are analogues of Branson's Q-curvature (of [7, 5]) and a host new conformal invariants. The work [1] of Aubry-Guillarmou shows that these objects play a deep role in geometric scattering, and one that generalises the corresponding results for the GJMS operators and Q-curvature; see also [26]. In these constructions, the null space (or kernel) of the operators L^k_ℓ is important.

Except at the lowest orders, explicit general formulae are not available for the L^k_ℓ. For any particular operator a formula may be obtained algorithmically via tractor calculus by the theory developed in [28, 29]. However it is clear from the special case of GJMS operators that the resulting operators, when presented in the usual way, would be given by extremely complicated formulae.

It was shown by Graham [33, 20] using the Fefferman-Graham ambient metric, and via a new construction in [24], that for the GJMS operators striking simplifications are available on conformally Einstein manifolds. In particular very simple factorisation formulae are available which mean that in this setting the operators may be given by a formula which is no more complicated than Branson's corresponding formula of Branson for the case of the standard round sphere [5]. (We note here that similar formulae are available for the conformal "powers" of the Dirac operator on the sphere [38, 18].) This means that in this setting (and without restriction on signature or compactness) one may obtain explicit decompositions of the null space of the GJMS operators in terms of eigenspaces of the Laplacian [31].

The situation is considerably more subtle for the operators L^k_ℓ on differential forms. Nevertheless in [32] it was shown that in setting of an Einstein structure one

may recover rather directly the detour complexes and Q-operators on differential forms. What is more these were shown to be given by simple explicit formulae; these involved factorisations of the operators which generalise those from [24]. That work [32] showed that for the case of $w = 0$ (i.e. when the domain space is that of *true* or *unweighted* differential forms) it is possible to treat the operators L^k_ℓ in a manner very similar to the treatment of the GJMS operators in [24]. It is not so for the remaining weights $w \neq 0$.

The aim of this article is to complete the picture by treating the operators L^k_ℓ, specialised to the setting of an Einstein structure and when $w \neq 0$. In fact we do more than this in that we obtain an approach which enables us to deal with all weights. Thus we we recover and significantly extend many of the results of [32].

We shall show that operators L^ℓ_k are compositions $L^\ell_k = S_1 \dots S_\ell$ where every factor on the right hand side has the form $S_1 = ad\delta + b\delta d + c$ for some scalars $a, b, c \in \mathbb{R}$. (Note these 2nd order factors commute.) That is, the operators L^ℓ_k are polynomials in $d\delta$ and δd. In detail, we have the following theorem which completely describes the differential operators L^ℓ_k on Einstein manifolds.

Theorem 1.1. *Let $\Phi := \{1, \dots, \ell\}$ and $w = k + l - n/2$ where $1 \leq k \leq \frac{n}{2}$. The operator*

$$L^\ell_k : \mathcal{E}^k[w] \to \mathcal{E}^k[w - 2\ell]$$

has the explicit form

$$L^\ell_k \sim \begin{cases} (d\delta - \delta d) P_k^{\Phi \setminus \{\ell\}}(d\delta, \delta d) & k = n/2 \\ P_k^{\Phi}(d\delta, \delta d) & w \leq 0 \quad \text{and} \quad k < n/2 \\ \widetilde{P}_k(d\delta, \delta d) P_k^{\Phi \setminus \{w, w+1\}}(d\delta, \delta d) & w \geq 1 \quad \text{and} \quad k < n/2 \end{cases},$$

for n even and

$$L^\ell_k \sim P_k^{\Phi}(d\delta, \delta d)$$

for n odd where $\sim$ means "is equal up to nonzero scalar multiple" and two variable polynomials P_k and $\widetilde{P}_k$ are given in (29) and (30).

Theorem 5.2 provides the main step needed to reach the Theorem above. Note that the former is interesting and important in its own right.

In [31] we show that if the factors $S_i : \mathcal{V} \to \mathcal{V}$ (for some vector space $\mathcal{V}$), in a composition $P := S_0 S_1 \cdots S_\ell$ of mutually commuting operators, are suitably *relatively invertible*, then the general inhomogeneous problem $Pu = f$ decomposes into an equivalent system $S_i u_i = f$, $i = 0, \cdots, \ell$. For the factors of the operators L^ℓ_k a sufficient form of relative invertibility is established in Proposition 6.2, in the case that the Einstein manifold is not Ricci flat. This is then used to reduce the generally high order conformal operators L^ℓ_k to equivalent lower order systems. The outcome is that in any signature (and without any assumption of compactness) on non-Ricci flat Einstein manifolds we can describe the spaces $\mathcal{N}(L^\ell_k)$ (the null space of L^ℓ_k), explicitly as a direct sum of null spaces for the second order factors of the L^ℓ_k (as in the Theorem above). This is Theorem 6.3. In the case of Riemannian signature compact manifolds of course the Hodge decomposition may be used in conjunction with the Theorem 1.1, and so we obtain Theorem 6.1.

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2. Background: Einstein metrics and conformal geometry

Let M be a smooth manifold, equipped with a Riemannian metric g_{ab}. Here and throughout we employ Penrose's abstract index notation. We shall write $\mathcal{E}^a$ to denote the space of smooth sections of the tangent bundle TM on M, and $\mathcal{E}_a$ for the space of smooth sections of the cotangent bundle T^*M. (In fact we will often use the same symbols for the bundles themselves.) We write $\mathcal{E}$ for the space of smooth functions and all tensors considered will be assumed smooth without further comment. An index which appears twice, once raised and once lowered, indicates a contraction. For simplicity we shall assume that the manifold M has dimension $n \geq 3$.

We first sketch here notation and background for general conformal structures and their tractor calculus following [15, 28]. Recall that a *conformal structure* of signature (p,q) on M is an equivalence class c of metrics, where the equivalence relation $g \sim \widehat{g}$ of metrics in c is that $\widehat{g} = fg$ for some positive function f. Equivalently a conformal structure is a smooth ray subbundle $\mathcal{Q} \subset S^2T^*M$ whose fibre over x consists of conformally related signature-(p,q) metrics at the point x. Sections of $\mathcal{Q}$ are metrics g on M. The principal bundle $\pi : \mathcal{Q} \to M$ has structure group $\mathbb{R}_+$, and each representation $\mathbb{R}_+ \ni x \mapsto x^{-w/2} \in \mathrm{End}(\mathbb{R})$ induces a natural line bundle on $(M,[g])$ that we term the conformal density bundle $E[w]$. We shall write $\mathcal{E}[w]$ for the space of sections of this bundle. Given a vector bundle V (or its space of sections $\mathcal{V}$) we shall write $V[w]$ (resp. $\mathcal{V}[w]$) to mean $V \otimes E[w]$ (resp. $\mathcal{V} \otimes \mathcal{E}[w]$). Here and throughout, sections, tensors, and functions are always smooth, meaning C^∞. When no confusion is likely to arise, we will use the same notation for a bundle and its section space.

We write $\boldsymbol{g}$ for the *conformal metric*, that is the tautological section of $S^2T^*M \otimes E[2]$ determined by the conformal structure. This will be used to identify TM with $T^*M[2]$. For many calculations we will use abstract indices in an obvious way. Given a choice of metric g from the conformal class, we write ∇ for the corresponding Levi-Civita connection. With these conventions the Laplacian Δ is given by $\Delta = \boldsymbol{g}^{ab}\nabla_a\nabla_b = \nabla^b\nabla_b$. Note $E[w]$ is trivialised by a choice of metric g from the conformal class, and we write ∇ for the connection arising from this trivialisation. It follows immediately that (the coupled) ∇_a preserves the conformal metric.

Since the Levi-Civita connection is torsion-free, the (Riemannian) curvature $R_{ab}{}^c{}_d$ is given by $[\nabla_a, \nabla_b]v^c = R_{ab}{}^c{}_d v^d$ where $[\cdot,\cdot]$ indicates the commutator bracket. The Riemannian curvature can be decomposed into the totally trace-free Weyl curvature C_{abcd} and a remaining part described by the symmetric *Schouten tensor* P_{ab}, according to $R_{abcd} = C_{abcd} + 2\boldsymbol{g}_{c[a}P_{b]d} + 2\boldsymbol{g}_{d[b}P_{a]c}$, where $[\cdots]$ indicates antisymmetrisation over the enclosed indices. We put $J := P^a{}_a$. The *Cotton tensor* is defined by

$$A_{abc} := 2\nabla_{[b}P_{c]a}.$$

Under a *conformal transformation* we replace a choice of metric g by the metric $\hat{g} = e^{2\omega} g$, where ω is a smooth function. Explicit formulae for the corresponding transformation of the Levi-Civita connection and its curvatures are given in e.g. [28]. We recall that, in particular, the Weyl curvature is conformally invariant $\widehat{C}_{abcd} = C_{abcd}$.

2.1. **Conformally invariant operators on weighted forms.** Following [6] we shall write $\mathcal{E}^k$ to denote the section space of smooth k-forms and $\mathcal{E}^k[w] = \mathcal{E}^k \otimes \mathcal{E}[w]$. Although this is similar to the notation for (weighted) tangent sections, by context no confusion should arise.

The invariant differential operators on conformally flat manifolds are all known. We shall refer to the summary of the classification in [17, section 3]. In particular, on weighted k-forms $\mathcal{E}^k[w]$ there are three types of operators. Here we shall mainly focus on the (power) Laplacian like operators

$$L_k^\ell : \mathcal{E}^k[w] \to \mathcal{E}^k[w - 2\ell], \qquad k \leq \frac{n}{2}, \ w = k + \ell - n/2.$$

Here n is the dimension and $\ell \geq 1$ the order, i.e. $w \geq k - \frac{n}{2} + 1$. That is, w is an integer for n even and a half integer for n odd. Using the terminology of [17, section 3], these operators are all non–standard for n odd and, assuming n even, they are regular for $w \geq k + 1$ and $w = 0$ and singular in remaining cases, i.e. for $w \in \{k - \frac{n}{2} + 1, \ldots, k\} \setminus \{0\}$. Further possible operators are the exterior derivative d and its formal adjoint δ

$$d : \mathcal{E}^k[0] \to \mathcal{E}^{k+1}[0], \ k \leq n-1 \quad \text{and} \quad \delta : \mathcal{E}^k[-n+2k] \to \mathcal{E}^{k-1}[-n+2k-2], \ k \geq 1$$

of differential order 1. We extend the use of this notation to weighted differential forms in the obvious way via the Levi-Civita connection; for $f \in \mathcal{E}^k[w]$ we write df and δf to mean

$$(k+1)\nabla_{[a_0} f_{a_1 \cdots a_k]} \quad \text{and} \quad -\nabla^{a_1} f_{a_1 \cdots a_k},$$

respectively.

Finally, for integers $w \geq k + 1$, $k \geq 1$ and $w' \geq 1$ there are overdetermined operators

$$\mathcal{E}^k[w] \to \underbrace{\mathcal{E}^1 \otimes \ldots \otimes \mathcal{E}^1}_{w-k} \otimes \mathcal{E}^k[w] \quad \text{and} \quad \mathcal{E}[w'] \to \underbrace{\mathcal{E}^1 \otimes \ldots \otimes \mathcal{E}^1}_{w'+1}[w']$$

of differential order $w - k$ and $w' + 1$, respectively. More precisely, the target bundle is the Cartan component of the displayed space. These operators are regular and are a class of what are known as first BGG operators.

2.2. **Conformal geometry and tractor calculus.** A central tool in the treatment of conformal geometry is tractor calculus [2], since this is a conformally invariant replacement of the Ricci calculus of pseudo-Riemannian geometry. (For a general development of tractor calculus in the broader context of all parabolic geometries see [9]). The discussion here follows [28] and for the treatment of forms [6, 30, 45] as summarised in [32]. Some parts of the treatment are specialised to Einstein manifolds.

We first recall the definition of the standard tractor bundle over $(M,[g])$. This is a vector bundle of rank $n+2$ defined, for each $g \in [g]$, by $[\mathcal{E}^A]_g = \mathcal{E}[1] \oplus \mathcal{E}_a[1] \oplus \mathcal{E}[-1]$. If $\widehat{g} = e^{2\Upsilon} g$, we identify $(\alpha, \mu_a, \tau) \in [\mathcal{E}^A]_g$ with $(\widehat{\alpha}, \widehat{\mu}_a, \widehat{\tau}) \in [\mathcal{E}^A]_{\widehat{g}}$ by the transformation

$$(2) \qquad \begin{pmatrix} \widehat{\alpha} \\ \widehat{\mu}_a \\ \widehat{\tau} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \Upsilon_a & \delta_a{}^b & 0 \\ -\frac{1}{2}\Upsilon_c\Upsilon^c & -\Upsilon^b & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \mu_b \\ \tau \end{pmatrix},$$

where $\Upsilon_a := \nabla_a \Upsilon$. It is straightforward to verify that these identifications are consistent upon changing to a third metric from the conformal class, and so taking the quotient by this equivalence relation defines the *standard tractor bundle* $\mathcal{T}$, or $\mathcal{E}^A$ in an abstract index notation, over the conformal manifold. (Alternatively the standard tractor bundle may be constructed as a canonical quotient of a certain 2-jet bundle or as an associated bundle to the normal conformal Cartan bundle [14].) On a conformal structure of signature (p,q), the bundle $\mathcal{E}^A$ admits an invariant metric h_{AB} of signature $(p+1, q+1)$ and an invariant connection, which we shall also denote by ∇_a, preserving h_{AB}. In a conformal scale g, these are given by

$$(3) \qquad h_{AB} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \boldsymbol{g}_{ab} & 0 \\ 1 & 0 & 0 \end{pmatrix} \text{ and } \nabla_a \begin{pmatrix} \alpha \\ \mu_b \\ \tau \end{pmatrix} = \begin{pmatrix} \nabla_a \alpha - \mu_a \\ \nabla_a \mu_b + \boldsymbol{g}_{ab}\tau + P_{ab}\alpha \\ \nabla_a \tau - P_{ab}\mu^b \end{pmatrix}.$$

It is readily verified that both of these are conformally well-defined, i.e., independent of the choice of a metric $g \in [g]$. Note that h_{AB} defines a section of $\mathcal{E}_{AB} = \mathcal{E}_A \otimes \mathcal{E}_B$, where $\mathcal{E}_A$ is the dual bundle of $\mathcal{E}^A$. Hence we may use h_{AB} and its inverse h^{AB} to raise or lower indices of $\mathcal{E}_A$, $\mathcal{E}^A$ and their tensor products.

In computations, it is often useful to introduce the 'projectors' from $\mathcal{E}^A$ to the components $\mathcal{E}[1]$, $\mathcal{E}_a[1]$ and $\mathcal{E}[-1]$ which are determined by a choice of scale. They are respectively denoted by $X_A \in \mathcal{E}_A[1]$, $Z_{Aa} \in \mathcal{E}_{Aa}[1]$ and $Y_A \in \mathcal{E}_A[-1]$, where $\mathcal{E}_{Aa}[w] = \mathcal{E}_A \otimes \mathcal{E}_a \otimes \mathcal{E}[w]$, etc. Using the metrics h_{AB} and $\boldsymbol{g}_{ab}$ to raise indices, we define X^A, Z^{Aa}, Y^A. Then we immediately see that

$$Y_A X^A = 1, \quad Z_{Ab} Z^A{}_c = \boldsymbol{g}_{bc},$$

and that all other quadratic combinations that contract the tractor index vanish. In (2) note that $\widehat{\alpha} = \alpha$ and hence X^A is conformally invariant.

Given a choice of conformal scale, the *tractor-D operator* $D_A \colon \mathcal{E}_{B\cdots E}[w] \to \mathcal{E}_{AB\cdots E}[w-1]$ is defined by

$$(4) \qquad D_A V := (n+2w-2) w Y_A V + (n+2w-2) Z_{Aa} \nabla^a V - X_A \Box V,$$

where $\Box V := \Delta V + wJV$. This also turns out to be conformally invariant as can be checked directly using the formulae above (or alternatively there are conformally invariant constructions of D, see e.g. [22]).

The curvature Ω of the tractor connection is defined by

$$[\nabla_a, \nabla_b] V^C = \Omega_{ab}{}^C{}_E V^E$$

for $V^C \in \mathcal{E}^C$. Using (3) and the formulae for the Riemannian curvature yields

$$(5) \qquad \Omega_{abCE} = Z_C{}^c Z_E{}^e C_{abce} - 2X_{[C} Z_{E]}{}^e A_{eab}$$

We will also need a conformally invariant curvature quantity defined as follows (cf. [22, 23])

$$W_{BC}{}^{E}{}_{F} := \frac{3}{n-2} D^A X_{[A} \Omega_{BC]}{}^{E}{}_{F}, \tag{6}$$

where $\Omega_{BC}{}^{E}{}_{F} := Z_B{}^b Z_C{}^c \Omega_{bc}{}^{E}{}_{F}$. In a choice of conformal scale, W_{ABCE} is given by

$$\begin{aligned} &(n-4)\big(Z_A{}^a Z_B{}^b Z_C{}^c Z_E{}^e C_{abce} - 2 Z_A{}^a Z_B{}^b X_{[C} Z_{E]}{}^e A_{eab} \\ &-2 X_{[A} Z_{B]}{}^b Z_C{}^c Z_E{}^e A_{bce}\big) + 4 X_{[A} Z_{B]}{}^b X_{[C} Z_{E]}{}^e B_{eb}, \end{aligned} \tag{7}$$

where

$$B_{ab} := \nabla^c A_{acb} + P^{dc} C_{dacb}.$$

is known as the *Bach tensor*. From the formula (7) it is clear that W_{ABCD} has Weyl tensor type symmetries.

We will work with conformally Einstein manifolds. That is, conformal structures with an Einstein metric in the conformal class. This is the same as the existence of a non-vanishing section $\sigma \in \mathcal{E}[1]$ satisfying $\left[\nabla_{(a}\nabla_{b)_0} + P_{(ab)_0}\right]\sigma = 0$ where the subscript $(\ldots)_0$ indicates the trace-free symmetric part. Equivalently (see e.g. [2, 27]) there is a standard tractor I_A that is parallel with respect to the normal tractor connection ∇ and such that $\sigma := X_A I^A$ is non-vanishing. It follows that $I_A := \frac{1}{n} D_A \sigma = Y_A \sigma + Z_A^a \nabla_a \sigma - \frac{1}{n} X_A(\Delta + J)\sigma$, for some section $\sigma \in \mathcal{E}[1]$, and so $X^A I_A = \sigma$ is non-vanishing. If we compute in the scale σ, then the Cotton and Bach tensors are zero (see e.g. [27]) and so $W_{ABCD} = (n-4) Z_A^a Z_B^b Z_C^c Z_D^d C_{abcd}$.

2.3. **Tractor forms.** Here we recall the calculus for tractor forms as developed in [32]. We write $\mathcal{E}^k[w]$ for the space of sections of $(\Lambda^k T^* M) \otimes E[w]$ (and $\mathcal{E}^k = \mathcal{E}^k[0]$). Further we put $\mathcal{E}_k[w] := \mathcal{E}^k[w + 2k - n]$. We shall use the analogous notation $\mathcal{T}^k[w] := (\Lambda^k \mathcal{T}) \otimes \mathcal{E}[w]$ on the tractor level.

In order to be explicit and efficient in calculations involving bundles of possibly high rank it is necessary to employ abstract index notation as follows. In the usual abstract index conventions one would write $\mathcal{E}_{[ab\cdots c]}$ (where there are implicitly k-indices skewed over) for the space $\mathcal{E}^k$. To simplify subsequent expressions we use the following conventions. Firstly indices labelled with sequential superscripts which are at the same level (i.e. all contravariant or all covariant) will indicate a completely skew set of indices. Formally we set $a^1 \cdots a^k = [a^1 \cdots a^k]$ and so, for example, $\mathcal{E}_{a^1 \cdots a^k}$ is an alternative notation for $\mathcal{E}^k$ while $\mathcal{E}_{a^1 \cdots a^{k-1}}$ and $\mathcal{E}_{a^2 \cdots a^k}$ both denote $\mathcal{E}^{k-1}$. Next, following [30] we abbreviate this notation via multi-indices: We will use the forms indices

$$\begin{aligned} \mathbf{a}^k &:= a^1 \cdots a^k = [a^1 \cdots a^k], \quad k \geq 0, \\ \dot{\mathbf{a}}^k &:= a^2 \cdots a^k = [a^2 \cdots a^k], \quad k \geq 1. \end{aligned}$$

If $k = 1$ then $\dot{\mathbf{a}}^k$ simply means the index is absent. The corresponding notations will be used for tractor indices so e.g. the bundle of tractor k–forms $\mathcal{E}_{[A^1 \ldots A^k]}$ will be denoted by $\mathcal{E}_{A^1 \ldots A^k}$ or $\mathcal{E}_{\mathbf{A}^k}$.

The structure of $\mathcal{E}_{\mathbf{A}^k}$ is

$$\mathcal{E}_{[A^1 \ldots A^k]} = \mathcal{E}_{\mathbf{A}^k} \simeq \mathcal{E}^{k-1}[k] \oplus\!\!\!\!+ \left(\mathcal{E}^k[k] \oplus \mathcal{E}^{k-2}[k-2]\right) \oplus\!\!\!\!+ \mathcal{E}^{k-1}[k-2]; \tag{8}$$

in a choice of scale the semidirect sums $\oplus\!\!\!\!+$ may be replaced by direct sums and otherwise they indicate the composition series structure arising from the tensor powers of (2).

In a choice of metric g from the conformal class, the projectors (or splitting operators) X, Y, Z for $\mathcal{E}_A$ determine corresponding projectors $\mathbb{X}, \mathbb{Y}, \mathbb{Z}, \mathbb{W}$ for $\mathcal{E}_{\mathbf{A}^{k+1}}$, $k \geq 1$ These execute the splitting of this space into four components and are given as follows.

$$
\begin{array}{llllll}
\mathbb{Y}^k & = \mathbb{Y}_{A^0 A^1 \cdots A^k}{}^{a^1 \cdots a^k} & = \mathbb{Y}_{A^0 \mathbf{A}^k}{}^{\mathbf{a}^k} & = Y_{A^0} Z_{A^1}^{a^1} \cdots Z_{A^k}^{a^k} & \in \mathcal{E}_{\mathbf{A}^{k+1}}^{\mathbf{a}^k}[-k-1] \\
\mathbb{Z}^k & = \mathbb{Z}_{A^1 \cdots A^k}^{a^1 \cdots a^k} & = \mathbb{Z}_{\mathbf{A}^k}^{\mathbf{a}^k} & = Z_{A^1}^{a^1} \cdots Z_{A^k}^{a^k} & \in \mathcal{E}_{\mathbf{A}^k}^{\mathbf{a}^k}[-k] \\
\mathbb{W}^k & = \mathbb{W}_{A' A^0 A^1 \cdots A^k}{}^{a^1 \cdots a^k} & = \mathbb{W}_{A' A^0 \mathbf{A}^k}{}^{\mathbf{a}^k} & = X_{[A'} Y_{A^0} Z_{A^1}^{a^1} \cdots Z_{A^k]}^{a^k} & \in \mathcal{E}_{\mathbf{A}^{k+2}}^{\mathbf{a}^k}[-k] \\
\mathbb{X}^k & = \mathbb{X}_{A^0 A^1 \cdots A^k}{}^{a^1 \cdots a^k} & = \mathbb{X}_{A^0 \mathbf{A}^k}{}^{\mathbf{a}^k} & = X_{A^0} Z_{A^1}^{a^1} \cdots Z_{A^k}^{a^k} & \in \mathcal{E}_{\mathbf{A}^{k+1}}^{\mathbf{a}^k}[-k+1]
\end{array}
$$

where $k \geq 0$. The superscript k in $\mathbb{Y}^k$, $\mathbb{Z}^k$, $\mathbb{W}^k$ and $\mathbb{X}^k$ shows the corresponding tensor valence. (This is slightly different than in [6], where k is the relevant tractor valence.) Note that $Y = \mathbb{Y}^0$, $Z = \mathbb{Z}^1$ and $X = \mathbb{X}^0$ and $\mathbb{W}^0 = X_{[A'} Y_{A^0]}$. To simplify notation we introduce projectors/injectors

$$
\begin{aligned}
q^k : \mathcal{T}^k[w] &\to \mathcal{E}^k[w+k], \quad F_{\mathbf{A}} \mapsto \mathbb{Z}_{\mathbf{a}}^{\mathbf{A}} F_{\mathbf{A}} \quad \text{for} \quad F \in \mathcal{T}^k \quad \text{and} \\
q_k : \mathcal{E}^k[w] &\to \mathcal{T}^k[w-k], \quad f_{\mathbf{a}} \mapsto \mathbb{Z}_{\mathbf{A}}^{\mathbf{a}} f_{\mathbf{a}} \quad \text{for} \quad f \in \mathcal{E}^k.
\end{aligned}
$$

From (3) we immediately see $\nabla_p Y_A = Z_A^a P_{pa}$, $\nabla_p Z_A^a = -\delta_p^a Y_A - P_p^a X_A$ and $\nabla_p X_A = Z_{Ap}$. From this we obtain the formulae (cf. [30])

$$
\begin{aligned}
\nabla_p \mathbb{Y}_{A^0 \mathbf{A}^k}{}^{\mathbf{a}^k} &= P_{pa_0} \mathbb{Z}_{A^0 \mathbf{A}^k}^{a^0 \mathbf{a}^k} + k P_p{}^{a^1} \mathbb{W}_{A^0 \mathbf{A}^k}{}^{\dot{\mathbf{a}}^k} \\
\nabla_p \mathbb{Z}_{A^0 \mathbf{A}^k}^{a^0 \mathbf{a}^k} &= -(k+1) \delta_p^{a^0} \mathbb{Y}_{A^0 \mathbf{A}^k}{}^{\mathbf{a}^k} - (k+1) P_p{}^{a^0} \mathbb{X}_{A^0 \mathbf{A}^k}{}^{\mathbf{a}^k} \\
\nabla_p \mathbb{W}_{A^0 \mathbf{A}^k}{}^{\dot{\mathbf{a}}^k} &= -\boldsymbol{g}_{pa^1} \mathbb{Y}_{A^0 \mathbf{A}^k}{}^{\mathbf{a}^k} + P_{pa^1} \mathbb{X}_{A^0 \mathbf{A}^k}{}^{a^1 \dot{\mathbf{a}}^k} \\
\nabla_p \mathbb{X}_{A^0 \mathbf{A}^k}{}^{\mathbf{a}^k} &= \boldsymbol{g}_{pa^0} \mathbb{Z}_{A^0 \mathbf{A}^k}^{a^0 \mathbf{a}^k} - k \delta_p^{a^1} \mathbb{W}_{A^0 \mathbf{A}^k}{}^{\dot{\mathbf{a}}^k},
\end{aligned} \tag{9}
$$

which determine the tractor connection on form tractors in a conformal scale. Similarly, one can compute the Laplacian Δ applied to the tractors $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$ and $\mathbb{W}$. As an operator on form tractors we have the opportunity to modify Δ by adding some amount of $W\sharp\sharp$, where $\sharp$ denotes the natural tensorial action of sections in $\mathrm{End}(\mathcal{E}^A)$. Analogously, we shall use $C\sharp\sharp$ to modify the Laplacian on forms; here $\sharp$ denotes the natural tensorial action of sections in $\mathrm{End}(\mathcal{E}^a)$. It turns out (cf. [6]) that it will be convenient for us to use modifications

$$
\not{\Delta} = \Delta + \frac{1}{n-4} W\sharp\sharp \quad \text{and} \quad \not{D}_A = D_A - \frac{1}{n-4} X_A W\sharp\sharp \qquad \text{for} \quad n \neq 4, \tag{10}
$$

cf. (4). (Note $\Delta = \nabla^a \nabla_a$.) The operator $\not{D}$ was introduced in [6].

Since the Laplacian is of the second order, it is convenient to consider e.g. $\not{\Delta} \mathbb{Y}_{\mathbf{A}}^{\dot{\mathbf{a}}} \tau_{\dot{\mathbf{a}}}$ where $\tau_{\dot{\mathbf{a}}} \in \mathcal{E}_{\dot{\mathbf{a}}}[w]$. It will be sufficient for our purpose to calculate this only in an

Einstein scale. For example, using (9) and then that $P_{ab} = \boldsymbol{g}_{ab}J/n$, we have

$$
\begin{aligned}
\nabla^p\nabla_p \mathbb{Y}_{\mathbf{A}}^{\dot{\mathbf{a}}}\tau_{\dot{\mathbf{a}}} =& \nabla^p\big[P_{pa^1}\mathbb{Z}_{\mathbf{A}}^{\mathbf{a}} + (k-1)P_p{}^{a^2}\mathbb{W}_{\mathbf{A}}^{\ddot{\mathbf{a}}} + \mathbb{Y}_{\mathbf{A}}^{\dot{\mathbf{a}}}\nabla_p\big]\tau_{\dot{\mathbf{a}}} \\
=& -\mathbb{Y}_{\mathbf{A}}^{\dot{\mathbf{a}}}\Big[\big(\delta d + d\delta + (1-\frac{2(k-1)(n-k+1)}{n})J + C\sharp\sharp\big)\tau\Big]_{\dot{\mathbf{a}}} \\
&+ \frac{2}{nk}\mathbb{Z}_{\mathbf{A}}^{\mathbf{a}}(Jd\tau)_{\mathbf{a}} - \frac{2(k-1)}{n}\mathbb{W}_{\mathbf{A}}^{\ddot{\mathbf{a}}}(J\delta\tau)_{\ddot{\mathbf{a}}} - \frac{n-2k+2}{n^2}\mathbb{X}_{\mathbf{A}}^{\dot{\mathbf{a}}}J^2\tau_{\dot{\mathbf{a}}},
\end{aligned}
$$

where, as usual, $\mathbf{A} = \mathbf{A}^k$ and $\mathbf{a} = \mathbf{a}^k$. Summarising, one can compute that in an Einstein scale we obtain

$$
\begin{aligned}
-\not{\Delta}\mathbb{Y}_{\mathbf{A}}^{\dot{\mathbf{a}}}\tau_{\dot{\mathbf{a}}} &= \mathbb{Y}_{\mathbf{A}}^{\dot{\mathbf{a}}}\Big[\big(\delta d + d\delta + (1-\frac{2(k-1)(n-k+1)}{n})J\big)\tau\Big]_{\dot{\mathbf{a}}} \\
&\quad - \frac{2}{nk}\mathbb{Z}_{\mathbf{A}}^{\mathbf{a}}(Jd\tau)_{\mathbf{a}} + \frac{2(k-1)}{n}\mathbb{W}_{\mathbf{A}}^{\ddot{\mathbf{a}}}(J\delta\tau)_{\ddot{\mathbf{a}}} + \frac{n-2k+2}{n^2}\mathbb{X}_{\mathbf{A}}^{\dot{\mathbf{a}}}J^2\tau_{\dot{\mathbf{a}}} \\
-\not{\Delta}\mathbb{Z}_{\mathbf{A}}^{\mathbf{a}}\mu_{\mathbf{a}} &= -2k\mathbb{Y}_{\mathbf{A}}^{\dot{\mathbf{a}}}(\delta\mu)_{\dot{\mathbf{a}}} + \mathbb{Z}_{\mathbf{A}}^{\mathbf{a}}\Big[\big(\delta d + d\delta - \frac{2k(n-k-1)}{n}J\big)\mu\Big]_{\mathbf{a}} \\
&\quad - \frac{2k}{n}\mathbb{X}_{\mathbf{A}}^{\dot{\mathbf{a}}}(J\delta\mu)_{\dot{\mathbf{a}}} \\
-\not{\Delta}\mathbb{W}_{\mathbf{A}}^{\ddot{\mathbf{a}}}\nu_{\ddot{\mathbf{a}}} &= \frac{2}{k-1}\mathbb{Y}_{\mathbf{A}}^{\dot{\mathbf{a}}}(d\nu)_{\dot{\mathbf{a}}} + \mathbb{W}_{\mathbf{A}}^{\ddot{\mathbf{a}}}\Big[\big(\delta d + d\delta - \frac{2(k-3)(n-k+2)}{n}J\big)\nu\Big]_{\ddot{\mathbf{a}}} \\
&\quad - \frac{2}{n(k-1)}\mathbb{X}_{\mathbf{A}}^{\dot{\mathbf{a}}}(d\nu)_{\dot{\mathbf{a}}} \\
-\not{\Delta}\mathbb{X}_{\mathbf{A}}^{\dot{\mathbf{a}}}\rho_{\dot{\mathbf{a}}} &= (n-2k+2)\mathbb{Y}_{\mathbf{A}}^{\dot{\mathbf{a}}}\rho_{\dot{\mathbf{a}}} - 2(k-1)\mathbb{W}_{\mathbf{A}}^{\ddot{\mathbf{a}}}(\delta\rho)_{\ddot{\mathbf{a}}} \\
&\quad - \frac{2}{k}\mathbb{Z}_{\mathbf{A}}^{\mathbf{a}}(d\rho)_{\mathbf{a}} + \mathbb{X}_{\mathbf{A}}^{\dot{\mathbf{a}}}\Big[\big(\delta d + d\delta + (1-\frac{2(k-1)(n-k+1)}{n})J\big)\rho\Big]_{\dot{\mathbf{a}}}.
\end{aligned} \tag{11}
$$

for $n \neq 4$, cf. [45, (1.50)]. Here $\tau_{\dot{\mathbf{a}}} \in \mathcal{E}_{\dot{\mathbf{a}}}[w]$, $\mu_{\mathbf{a}} \in \mathcal{E}_{\mathbf{a}}[w]$, $\nu_{\ddot{\mathbf{a}}} \in \mathcal{E}_{\ddot{\mathbf{a}}}[w]$ and $\rho_{\dot{\mathbf{a}}} \in \mathcal{E}_{\dot{\mathbf{a}}}[w]$ where $\mathbf{a} = \mathbf{a}^k$, $k \geq 1$ and w is any conformal weight.

2.4. **Modified tractor D-operator on conformally Einstein manifolds.** The operator $\not{D}$ will be essential for our subsequent computation. It is defined above only for $n \neq 4$. Assuming M is a conformally Einstein manifold, we extend this operator to the dimension $n = 4$ as follows. First we define the tractor

$$
\begin{aligned}
\widetilde{W}^{\sigma}_{ABCD} =& \mathbb{Z}_{AB}^{\,a\,b}\mathbb{Z}_{CD}^{\,c\,d}C_{abcd} - 4\mathbb{Z}_{AB}^{\,a\,b}\mathbb{X}_{CD}^{\,\,\,d}\nabla_{[a}P_{b]d} \\
&- 4\mathbb{X}_{AB}^{\,\,\,b}\mathbb{Z}_{CD}^{\,c\,d}\nabla_{[c}P_{d]b} - 8\mathbb{X}_{AB}^{\,\,\,b}\mathbb{X}_{CD}^{\,\,\,d}\sigma^{-1}(\nabla_{[b}P_{a]d})\nabla^a\sigma
\end{aligned} \tag{12}
$$

in the scale $\sigma \in \mathcal{E}[1]$. This (scale dependent quantity) was used in [24, section 4] and it is shown there that $\widetilde{W}^{\sigma_1}_{ABCD} = \widetilde{W}^{\sigma_2}_{ABCD}$ for any pair of Einstein scales σ_1 and σ_2. Hence we can drop the superscript σ in conformally Einstein manifolds as $\widetilde{W}_{ABCD}$ is well defined on such structures.

Using $\widetilde{W}_{ABCD}$, we define new modifications

$$
\not{\Delta} = \Delta + \widetilde{W}\sharp\sharp \quad \text{and} \quad \not{D}_A = D_A - X_A\widetilde{W}\sharp\sharp. \tag{13}
$$

In particular, this definition covers the case $n = 4$. The two definitions of $\not{\Delta}$ and $\not{D}$ in (10) and (13) for $n \neq 4$ are consistent; the following lemma follows from [24, Section 4].

Lemma 2.1. *On conformally Einstein manifolds it holds $\widetilde{W}_{ABCD} = W_{ABCD}$ for $n \neq 4$. Therefore the operators $\not{D}$ and $\not{\Delta}$ defined in (13) agree, in dimension $n \neq 4$, with the operators denoted by same symbols in expression (10).*

To write explicitly the commutator $[\not{D}_A, \not{D}_B]$ on density valued tractor fields, we shall need the following operator introduced in [23]. Recall that sequentially labelled indices are assumed to be skew over, e.g. $A^1A^2 = [A^1A^2]$. We put

$$D_{A^1A^2} = -2(w\mathbb{W}_{A^1A^2} + \mathbb{X}_{A^1}{}^{a}_{A^2}\nabla_a) \tag{14}$$

Using this, one computes

$$[\not{D}_A, \not{D}_B] = (n + 2w - 2)\big[(n + 2w - 4)\widetilde{W}_{AB}\sharp - (D_{AB}\widetilde{W})\sharp\sharp\big]. \tag{15}$$

on any density valued tractor field. Since we can use the definition (10) for $\not{D}$, in the case $n \neq 4$, the previous display follows from [32, (13)] for such dimensions. A direct computation then verifies the case $n = 4$.

Lemma 2.2. *Let $I^A, \bar{I}^A \in \mathcal{E}^A$ be two parallel tractors. Then $I^A\bar{I}^B[\not{D}_A, \not{D}_B] = 0$ on any density valued tractor fields.*

Proof. Since $I^AW_{ABCD} = 0$, the case $n \neq 4$ follows from [32, (13) and Lemma 2.2(ii)]. Assume $n = 4$. Then $I^A\widetilde{W}_{ABCD} = 0$. Choosing an Einstein scale σ, this is easily verified using relations $\Omega_{abCD}I^D = 0$, $\nabla_{[a}P_{b]c} = 0$ and $\sigma^{-1}(\nabla_{[b}P_{a]d})\nabla^a\sigma = 0$, cf. the explicit formula of $\widetilde{W}_{ABCD}$ above. Further one easily verifies that [32, Proposition 2.1 (ii)] holds if we replace W_{ABCD} by $\widetilde{W}_{ABCD}$. Therefore also [32, Lemma 2.2(ii)] holds if we replace W_{ABCD} by $\widetilde{W}_{ABCD}$ and the case $n = 4$ follows. □

3. Einstein manifolds: conformal Laplacian operators on tractors

We assume that the structure $(M, [g])$ is conformally Einstein, and write $\sigma \in \mathcal{E}[1]$ for some Einstein scale from the conformal class. Then $I^A := \frac{1}{n}D^A\sigma$ is parallel and $X^AI_A = \sigma$ is non-vanishing.

The operator $\not{\Box} := \Delta + wJ + \widetilde{W}\sharp\sharp$ acting on tractor bundles of the weight w is conformally invariant only if $n + 2w - 2 = 0$. On the other hand the scale σ (or equivalently I^A), yields the operator

$$\not{\Box}_\sigma := I^A\not{D}_A = \sigma(-\not{\Delta} - 2\frac{w}{n}(n + w - 1)J) : \mathcal{E}_{B\cdots E}[w] \longrightarrow \mathcal{E}_{B\cdots E}[w - 1] \tag{16}$$

which is well defined for any w, cf. [24]. Thus we can consider the composition $(\not{\Box}_\sigma)^p$, $p \in \mathbb{N}$ and we set $(\not{\Box}_\sigma)^0 := \mathrm{id}$. These operators generally depend on the choice of the scale σ but one has the following modification of [24, Theorem 3.1].

Theorem 3.1. [30] *Let $\sigma, \bar\sigma$ be two Einstein scales in the conformal class and consider the operators*

$$\frac{1}{\sigma^p}(\not\Box_\sigma)^p, \frac{1}{\bar\sigma^p}(\not\Box_{\bar\sigma})^p : \mathcal{E}_{B\cdots E}[w] \longrightarrow \mathcal{E}_{B\cdots E}[w-2p],$$

for $p \in \mathbb{Z}_{\geq 0}$. If $w = p - n/2$ then $\frac{1}{\sigma^p}(\not\Box_\sigma)^p = \frac{1}{\bar\sigma^p}(\not\Box_{\bar\sigma})^p$.

Proof. The proof is completely analogous to the proof of [32, Theorem 3.1] once we know $I^A \bar I^B [\not{D}_A, \not{D}_B] = 0$. Hence the theorem follows using Lemma 2.2. □

One can generalize the tractor–D operator to weighted forms as follows. First some notation. Exterior and interior multiplication by a tractor 1-form ω are given by

$$(17)\qquad \begin{aligned} (\varepsilon(\omega)\varphi)_{A_0\cdots A_k} &= (k+1)\omega_{[A_0}\varphi_{A_1\cdots A_k]}\,, \\ (\iota(\omega)\varphi)_{A_2\cdots A_k} &= \omega^{A_1}\varphi_{A_1\cdots A_k}\,. \end{aligned}$$

We extend the notation for interior and exterior multiplication in an obvious way to operators which increase the rank by one. For example, for φ a weighted tractor form, $\iota(\not{D})\varphi$ means $\not{D}^{A_1}\varphi_{A_1\cdots A_k}$.

The operators $M^{\mathbf{a}}_{\mathbf{A}}[w] : \mathcal{E}_{\mathbf{a}} \to \mathcal{E}_{\mathbf{A}}[w-k]$ (see the operator $\overline{M}$ from [30]) and $M^{*\mathbf{A}}_{\ \mathbf{a}} : \mathcal{E}_{\mathbf{A}}[w'] \to \mathcal{E}_{\mathbf{a}}[w'+k]$ defined as

$$(18)\qquad \begin{aligned} M^{\mathbf{a}}_{\mathbf{A}} f_{\mathbf{a}} &= \frac{n+w-2k}{k}\mathbb{Z}^{\mathbf{a}}_{\mathbf{A}} f_{\mathbf{a}} + \mathbb{X}^{\dot{\mathbf{a}}}_{\mathbf{A}}(\delta f)_{\dot{\mathbf{a}}}, \quad f_{\mathbf{a}} \in \mathcal{E}_{\mathbf{a}}[w] \\ M^{*\mathbf{A}}_{\ \mathbf{a}} F_{\mathbf{A}} &= -(w'+k)\mathbb{Z}^{\mathbf{A}}_{\mathbf{a}} F_{\mathbf{A}} + (d\mathbb{X}^{\mathbf{A}} F_{\mathbf{A}})_{\mathbf{a}}, \quad F_{\mathbf{A}} \in \mathcal{E}_{\mathbf{A}}[w']. \end{aligned}$$

where $\mathbf{A} = \mathbf{A}^k$ and $\mathbf{a} = \mathbf{a}^k$ are formal adjoints for suitable choice of w'. (Hence M^* is conformally invariant.) These operators are closely related to $\iota(\not{D})\varepsilon(X)$ and $\iota(X)\varepsilon(\not{D})$. One easily computes that

$$\begin{aligned} k(n+2(w-k)+2)Mf &= \iota(\not{D})\varepsilon(X)q_k f, \quad f \in \mathcal{E}^k[w] \\ -(n+2w'-2)M^*F &= q^k\iota(X)\varepsilon(\not{D})F, \quad F \in \mathcal{T}^k[w']. \end{aligned}$$

It follows that on differential forms of generic weight

$$(19)\qquad \iota(\not{D})M = 0.$$

In fact by computing the remaining case it follows that this holds for all weights. Another consequence for f and F as above we have

$$(20)\qquad \begin{aligned} &\iota(\not{D})\varepsilon(X)\iota(X)\varepsilon(\not{D})F = -k(n+2w'+2)(n+2w'-2)MM^*F, \\ &q^k\iota(X)\varepsilon(\not{D})\iota(\not{D})\varepsilon(X)q_k f = -k(n+2(w-k)+2)(n+2(w-k)-2)M^*Mf, \\ &\text{and } M^*Mf = -\frac{1}{k}w(n+w-2k)\,\mathrm{id}\,. \end{aligned}$$

Note that contrary to the last relation, MM^* is generally not a multiple of the identity.

Using these we obtain the (conformally invariant) operator

$$(21)\qquad \begin{aligned} \widetilde{\not{D}}_B &: f_{\mathbf{a}}[w] \longrightarrow f_{\mathbf{a}}[w-1] \\ \widetilde{\not{D}}_B f_{\mathbf{a}} &= M^{*\mathbf{A}}_{\ \mathbf{a}}\not{D}_B M^{\mathbf{a}}_{\mathbf{A}} f_{\mathbf{a}}. \end{aligned}$$

As an analogy of (16) we have (scale dependent) operators

$$\widetilde{\not\Box}_\sigma^{(p)} := M^{*}{}^{\mathbf{A}}_{\mathbf{a}}(\not\Box_\sigma)^p M^{\mathbf{a}}_{\mathbf{A}} : \mathcal{E}_{\mathbf{a}}[w] \longrightarrow \mathcal{E}_{\mathbf{a}}[w-p] \tag{22}$$

for $p \geq 1$ and we put $\widetilde{\not\Box}_\sigma^{(0)} := \mathrm{id}$. The case $p = 1$ shall be denoted simply as $\widetilde{\not\Box}_\sigma := \widetilde{\not\Box}_\sigma^{(1)}$. Note the operators $(\widetilde{\not\Box}_\sigma)^p$ and $\widetilde{\not\Box}_\sigma^{(p)}$ are generally different for $p \geq 2$. Using the formulae (4), (9), and (11) (and (12) for dimension 4) in a computation we obtain

$$\begin{aligned}\widetilde{\not\Box}_\sigma^{(1)} = \widetilde{\not\Box}_\sigma = -\frac{1}{k}\sigma\big[&w(n+w-2k-1)d\delta + (w-1)(n+w-2k)\delta d \\ &- 2\frac{w(w-1)}{n}(n+w-2k)(n+w-2k-1)J\big].\end{aligned} \tag{23}$$

and

$$\begin{aligned}\widetilde{\not\Box}_\sigma^{(2)} = &-\frac{1}{k}\sigma^2\big[w(n+w-2k-2)(d\delta)^2 + (w-2)(n+w-2k)(\delta d)^2 \\ &- \frac{2}{n}w(n+w-2k-2)[(w-1)(n+w-2k)+(w-2)(n+w-2k-1)]Jd\delta \\ &- \frac{2}{n}(w-2)(n+w-2k)[(w-1)(n+w-2k-2)+w(n+w-2k-1)]J\delta d \\ &+ \frac{4}{n^2}w(w-1)(w-2)(n+w-2k)(n+w-2k-1)(n+w-2k-2)J^2\big].\end{aligned}$$

Actually one computes $(\widetilde{\not\Box}_\sigma)^2 = -\frac{1}{k}(w-1)(n+w-2k-1)\widetilde{\not\Box}_\sigma^{(2)}$ from last two displays. This shows that for $w \notin \{1, -n+2k+1\}$, $\widetilde{\not\Box}_\sigma^{(2)}$ can be always decomposed into simpler factors. Later we shall need the case $w = 1$ and remarkably, this can be also decomposed. On $\mathcal{E}^k[1]$ we have

$$\widetilde{\not\Box}_\sigma^{(2)} = -\frac{2}{k}\sigma^2\Big[(\frac{n}{2}-k-\frac{1}{2})d\delta + (\frac{n}{2}-k+\frac{1}{2})\delta d\Big]\Big[d\delta - \delta d + \frac{4}{n}(\frac{n}{2}-k)J\Big] \tag{24}$$

where the right hand side is not a scalar multiple of $(\widetilde{\not\Box}_\sigma)^2$.

4. Conformal operators on weighted forms

Henceforth we assume $1 \leq k \leq \frac{n}{2}$ and the dimension n can be either odd or even. A point of notation, we shall use $\varphi \sim \psi$ for differential operators φ and ψ if they are equal up to a nonzero scalar multiple. Further we assume $\sigma \in \mathcal{E}[1]$ is an Einstein scale with the parallel tractor I^A.

Our main aim is to study the operator

$$L_k^\ell := \sigma^{-\ell} q^k (\not\Box_\sigma)^\ell M : \mathcal{E}^k[w] \longrightarrow \mathcal{E}^k[w-2\ell], \quad w := k+\ell-n/2, \tag{25}$$

for $\ell \geq 0$. Note that the q^k projection on the left hand side is invariant since $\iota(X)(\not\Box_\sigma)^\ell M = 0$ for w as in the previous display, this follows from (19), and Proposition 7.3 with [6, Lemma 4.2]. It follows from Theorem 3.1 that L_k^ℓ is independent on the choice of Einstein scale σ, and hence is *canonical* on conformally Einstein manifolds. (Note L_k^ℓ coincides with $\widetilde{\not\Box}_\sigma^{(\ell)}$ up to a scalar multiple.) The agreement of the L_k^ℓ here with the operators given by the same notation in [6] is

immediate from the construction in that source, which uses the fefferman-Graham, ambient metric, along with Proposition 7.3 from Section 7.

It is also interesting to understand the meaning of the "bottom slot" of $\sigma^{-\ell}(\not\Box_\sigma)^\ell Mf$ for $f \in \mathcal{E}^k[w]$, i.e. the operator

$$G_k^{\ell,\sigma} := \sigma^{-\ell} q^{k-1}\iota(Y)(\not\Box_\sigma)^\ell M : \mathcal{E}^k[w] \longrightarrow \mathcal{E}^k[w-2\ell-2], \quad w := k+\ell-n/2, \tag{26}$$

which depends on the choice of the Einstein scale σ.

The operators L_k^1 and $G_k^{1,\sigma}$ are particularly simple:

Theorem 4.1. *Let $w = k+1-n/2$. The operators*

$$L_k^1 : \mathcal{E}^k[w] \to \mathcal{E}^k[w-2] \quad \textit{and} \quad G_k^{1,\sigma} : \mathcal{E}^k[w] \to \mathcal{E}^{k-1}[w-4]$$

have explicit form

$$L_k^1 = \big(\frac{n}{2}-k-1\big)d\delta + \big(\frac{n}{2}-k+1\big)\delta d + \frac{2}{n}\big(\frac{n}{2}-k-1\big)\big(\frac{n}{2}-k+1\big)\big(\frac{n}{2}-k\big)J$$

$$G_k^{1,\sigma} = \delta\big[d\delta + \frac{2}{n}\big(\frac{n}{2}-k+1\big)\big(\frac{n}{2}-k\big)J\big].$$

Note that up to a nonzero scalar multiple, $L_{n/2}^1$ simplifies to $d\delta - \delta d$. On the other hand, there is the relation $G_k^{1,\sigma} = \frac{1}{n/2-k-1}\delta L_k^1$ for $\frac{n}{2}-k-1 \neq 0$. The latter conditions excludes true forms, i.e. the weight $w = 0$.

Proof. The theorem follows by a direct computation. Concerning L_k^1 for $k \neq \frac{n}{2}$ one can proceed also by the following (simpler) way. We use once again that Proposition 7.3 with [6, Lemma 4.2] implies that $\iota(X)\not\Box_\sigma M = 0$ on $\mathcal{E}^k[w]$. Thus $M^*\not\Box_\sigma M = -(w'+k)q^k\not\Box_\sigma M$ on $\mathcal{E}^k[w]$, cf. (18), where $w' = w-1$. Therefore comparing $L_k^1 = \sigma^{-\ell}q^k(\not\Box_\sigma)^\ell M$ with $\frac{1}{\sigma}\widetilde{\not\Box}_\sigma = \frac{1}{\sigma}M^*\not\Box_\sigma M$, we see these coincide up to a constant multiple for $w'-1 \neq 0$, i.e. for $k \neq \frac{n}{2}$. Thus the form of L_k^1 follows from (23) in the latter case. □

Before studying operators L_k^ℓ in detail, we describe their relation with the operators $G_k^{\ell,\sigma}$.

Theorem 4.2. *Let $w = k+\ell-n/2$ where $1 \leq k \leq \frac{n}{2}$. Computing in an Einstein scale σ, operators $G_k^{\ell,\sigma} : \mathcal{E}^k[k+l-n/2] \longrightarrow \mathcal{E}^{k-1}[k-\ell-n/2-2]$ satisfy*

$$wG_k^{\ell,\sigma} = -\delta L_k^\ell,$$

$$G_k^{\ell,\sigma} = \frac{k-1}{k(n+w-2k+1)}\sigma^{-1}L_{k-1}^\ell\sigma\delta \quad \textit{for} \quad k \geq 2.$$

This in particular means operators $G_k^{\ell,\sigma}$ are not too interesting for $w \neq 0$. This is in strong contrast to the case $w = 0$, see [32] for details. Note also the denominator in the second display is always nonzero as $n+w-2k+1 = n+(k+\ell-n/2)-2k+1 = (\frac{n}{2}-k)+(\ell+1) \geq 1$.

Proof. Since $\iota(X)(\not\Box_\sigma)^\ell M = 0$ on $\mathcal{E}^k[w]$ (as explained above), and using definitions of L_k^ℓ and $G_k^{\ell,\sigma}$ in (25) and (26), these operators appear in the tractor field

$$F_{\mathbf{A}} := \sigma^{-\ell}\big((\not\Box_\sigma)^\ell Mf\big)_{\mathbf{A}} = \mathbb{Z}_{\mathbf{A}}^{\mathbf{a}}(L_k^\ell f)_{\mathbf{a}} + k\mathbb{X}_{\mathbf{A}}^{\dot{\mathbf{a}}}(G_k^{\ell,\sigma}f)_{\dot{\mathbf{a}}} \in \mathcal{E}_{\mathbf{A}}[-\frac{n}{2}-\ell] \tag{27}$$

for $f \in \mathcal{E}_{\mathbf{a}}$. Here we use form abstract indices $\mathbf{A} = \mathbf{A}^k$ and $\mathbf{a} = \mathbf{a}^k$ as above. The weight on the right hand side is obtained as $w - k - 2\ell = -\frac{n}{2} - \ell$. This tractor form has the property

$$\iota(\not{D})F = \iota(\not{D})\sigma^{-\ell}(\not{\Box}_\sigma)^\ell Mf = \sigma^{-\ell}(\not{\Box}_\sigma)^{\ell+1}\iota(X)Mf = 0$$

according to Proposition 7.3 with [6, Lemma 4.2]. It remains to evaluate $0 = (\iota(\not{D})F)_{\dot{\mathbf{A}}} = -\not{D}^B F_{B\dot{\mathbf{A}}}$ in detail. Using the form of F from (27), one gets

$$\not{D}^B F_{B\dot{\mathbf{A}}} = 2\ell \mathbb{Z}_{\dot{\mathbf{A}}}^{\dot{\mathbf{a}}}\big[\big(k+\ell-\frac{n}{2}\big)(G_k^{\ell,\sigma}f)_{\dot{\mathbf{a}}} + (\delta L_k^\ell f)_{\dot{\mathbf{a}}}\big] + \mathbb{X}_{\dot{\mathbf{A}}}^{\ddot{\mathbf{a}}}\nu_{\ddot{\mathbf{a}}}$$

for some section $\nu_{\ddot{\mathbf{a}}} \in \mathcal{E}_{\ddot{\mathbf{a}}}[-\frac{n}{2} - \ell + (k-4)]$. Since $w = k + \ell - \frac{n}{2}$ and the previous is zero, the first relation of the theorem follows.

To prove the second relation, observe that $\iota(Y)(\not{\Box}_\sigma)^\ell M = \sigma^{-1}\iota(I)(\not{\Box}_\sigma)^\ell M = \sigma^{-1}(\not{\Box}_\sigma)^\ell\iota(I)M$. Here and below we compute everything in the scale σ. Since $k\frac{n+w-2k+1}{k-1}\iota(I)M = M\sigma\delta$ using (18), where σ denotes the multiplication by σ on the right hand side, and $G_k^{\ell,\sigma} = \sigma^{-\ell}q^{k-1}\iota(Y)(\not{\Box}_\sigma)^\ell M$ we obtain

$$k\frac{n+w-2k+1}{k-1}G_k^{\ell,\sigma} = k\frac{n+w-2k+1}{k-1}\sigma^{-\ell}q^{k-1}\iota(Y)(\not{\Box}_\sigma)^\ell M = \sigma^{-1}L_{k-1}^\ell\sigma\delta.$$

and the theorem follows. □

5. Factorization of operators L_k^ℓ

Our aim is to find explicit expressions for L_k^ℓ in the general case $\ell \geq 1$. This also yields explicit form of $G_k^{\ell,\sigma}$ due to Theorem 4.2. As above we assume the manifold is conformally Einstein, $\sigma \in \mathcal{E}[1]$ is an Einstein scale, and we write I^A to denote the parallel tractor corresponding to σ. First we prove the following result.

Lemma 5.1. *Assume $V \in \mathcal{T}^k[\ell_0 - n/2]$ where $\ell_0 \in \{0, 1, \cdots\}$ satisfies $\iota(X)(\not{\Box}_\sigma)^{\ell_0}V = 0$. Then*

$$M^*Mq^k\sigma^{-\ell_0}(\not{\Box}_\sigma)^{\ell_0}V = q^k\sigma^{-\ell_0}(\not{\Box}_\sigma)^{\ell_0}MM^*V.$$

Proof. The case $\ell_0 = 0$ is an easy computation using (18) and (20). We shall prove the case $\ell_0 = 1$ and $\ell_0 \geq 2$ separately.

(i) Assume $\ell_0 = 1$. We use the direct computation. First we shall use the assumption on

$$V_{\mathbf{A}} = \mathbb{Y}_{\mathbf{A}}^{\dot{\mathbf{a}}}\kappa_{\dot{\mathbf{a}}} + \mathbb{Z}_{\mathbf{A}}^{\mathbf{a}}\mu_{\mathbf{A}} + \mathbb{W}_{\mathbf{A}}^{\ddot{\mathbf{a}}}\nu_{\ddot{\mathbf{a}}} + \mathbb{X}_{\mathbf{A}}^{\dot{\mathbf{a}}}\rho_{\dot{\mathbf{a}}} \in \mathcal{E}_{\mathbf{A}}[1 - n/2],$$

i.e. that $\iota(X)(\not{\Box}_\sigma)^{\ell_0}V = \iota(X)\sigma(-\not{\Delta} + \frac{n-2}{2}J)V = 0$. Here we have used the formula (16). We shall need only the top slot this tractor, i.e.

$$\Big[\big(d\delta+\delta d+(1-\frac{2(k-1)(n-k+1)}{n}+\frac{n-2}{2})J\big)\kappa-2k(\delta\mu)+(n-2k+2)\rho+\frac{2}{k-1}d\nu\Big]_{\dot{\mathbf{a}}}$$

(using (11)); by our assumption this is zero. Applying the differential d to the last display we obtain

$$\Big[\big(d\delta + (\frac{n}{2} - \frac{2(k-1)(n-k+1)}{n})J\big)d\kappa - 2kd\delta\mu + 2(\frac{n}{2} - k + 1)d\rho\Big]_{\dot{\mathbf{a}}} = 0. \tag{28}$$

To prove the Lemma we compare both sides of the claimed equality. To compute the right hand side we first need that

$$(MM^*V)_{\mathbf{A}} = \frac{1}{k}(\frac{n}{2}-k+1)\mathbb{Z}_{\mathbf{A}}^{\mathbf{a}}[(\frac{n}{2}-k-1)\mu+\frac{1}{k}d\kappa]_{\mathbf{a}}+\mathbb{X}_{\mathbf{A}}^{\dot{\mathbf{a}}}[(\frac{n}{2}-k-1)\delta\mu+\frac{1}{k}\delta d\kappa]_{\dot{\mathbf{a}}}$$

using (18). Applying $q^k\sigma^{-1}\not\Box_\sigma = q^k(-\not\Delta+\frac{n+2}{2}J)$ to the last display we see that the right hand side $q^k\sigma^{-1}(\not\Box_\sigma)MM^*V$ is equal to

$$\frac{1}{k}(\frac{n}{2}-k+1)(\frac{n}{2}-k-1)\big[d\delta+\delta d-\frac{2k(n-k-1)}{n}J+\frac{n-2}{2}J\big]\mu+$$
$$\frac{1}{k}(\frac{n}{2}-k-1)\big[\frac{1}{k}d\delta d\kappa-2d\delta\mu\big]+\frac{1}{k^2}(\frac{n}{2}-k+1)\big[-\frac{2k(n-k-1)}{n}+\frac{n-2}{2}\big]Jd\kappa.$$

using (11) after some computation. The computation for the left hand side is simpler and we obtain that $M^*Mq^k\sigma^{-1}(\not\Box_\sigma)V$ is equal to

$$\frac{1}{k}(\frac{n}{2}-k+1)(\frac{n}{2}-k-1)\big[\big(d\delta+\delta d-\frac{2k(n-k-1)}{n}J+\frac{n-2}{2}J\big)\mu-\frac{2}{nk}Jd\kappa-\frac{2}{k}d\rho\big]_{\mathbf{a}}$$

using (11) and (20). Now a short computation reveals that the difference of the last two displays vanishes due to (28).

(ii) Now assume $\ell_0\geq 2$. Using once again Proposition 7.3 with [6, Lemma 4.2] we have

$$\iota(X)\varepsilon(\not D)\iota(\not D)\varepsilon(X)\sigma^{-\ell_0}(\not\Box_\sigma)^{\ell_0}V = \sigma^{-\ell_0}(\not\Box_\sigma)^{\ell_0}\varepsilon(\not D)\varepsilon(X)\iota(X)\varepsilon(\not D)V.$$

The left hand side is equal to $\iota(X)\varepsilon(\not D)\iota(\not D)\varepsilon(X)q_kq^k\sigma^{-\ell_0}(\not\Box_\sigma)^{\ell_0}V$ due to the assumption $\iota(X)(\not\Box_\sigma)^{\ell_0}V=0$. Therefore applying q^k to both sides of the previous display and using (20) we obtain

$$-4k(\ell_0-1)(\ell_0+1)M^*Mq^k\sigma^{-\ell_0}(\not\Box_\sigma)^{\ell_0}V = -4k(\ell_0+1)(\ell_0-1)q^k\sigma^{-\ell_0}(\not\Box_\sigma)^{\ell_0}MM^*V$$

since $V\in\mathcal{T}^k[\ell_0-n/2]$ and $q^k\sigma^{-\ell_0}(\not\Box_\sigma)^{\ell_0}V\in\mathcal{E}^k[k-\ell_0-n/2]$. The scalar factor can be omitted on both sides since we assume $k\geq 1$ and $\ell_0\geq 2$, and the Lemma follows. $\square$

The operator L_k^ℓ is defined using a power of $\not\Box_\sigma$. The following Theorem shows how to replace the factors $\not\Box_\sigma$ (which act on tractor forms) by factors $\widetilde{\not\Box}_\sigma^{(p)}$ (which act on tensor forms).

Theorem 5.2. *Let $1\leq p\leq \ell-1$. The operator $L_k^\ell:\mathcal{E}^k[k+l-n/2]\longrightarrow\mathcal{E}^k[k-\ell-n/2]$ satisfies*

$$\frac{1}{k}\big(k+(\ell-p)-n/2\big)\big(k-(\ell-p)-n/2\big)L_k^\ell = \sigma^{-p}L_k^{\ell-p}\widetilde{\not\Box}_\sigma^{(p)}.$$

Proof. Using (25) we get

$$L_k^\ell := q^k\sigma^{-\ell}(\not\Box_\sigma)^\ell M = \sigma^{-p}q^k\sigma^{-(\ell-p)}(\not\Box_\sigma)^{\ell-p}(\not\Box_\sigma)^pM$$

where, recall, $\iota(X)(\not\Box_\sigma)^{\ell-p}(\not\Box_\sigma)^pM=0$. Using the previous Lemma (with $\ell_0=\ell-p$) we have

$$\sigma^{-p}M^*Mq^k\sigma^{-(\ell-p)}(\not\Box_\sigma)^{\ell-p}(\not\Box_\sigma)^pM = \sigma^{-p}q^k\sigma^{-(\ell-p)}(\not\Box_\sigma)^{\ell-p}MM^*(\not\Box_\sigma)^pM.$$

Now using (20) on the left hand side and (22) on the right hand side we finally obtain

$$\frac{1}{k}\big(k+(\ell-p)-n/2\big)\big(k-(\ell-p)-n/2\big)q^k\sigma^{-\ell}(\not\Box_\sigma)^\ell M = \sigma^{-p}q^k\sigma^{-(\ell-p)}(\not\Box_\sigma)^{\ell-p}M\widetilde{\not\Box}^{(p)}_\sigma$$

and the Theorem follows. □

The crucial point is that we can use the theorem repeatedly to decompose L^ℓ_k, $\ell \geq 2$ into a composition of factors $\widetilde{\not\Box}^{(p)}_\sigma$. For most weights involved we can apply the theorem $(\ell-1)$ times with $p=1$. This yields a composition of second order factors, each of which is $\widetilde{\not\Box}_\sigma = \widetilde{\not\Box}^{(1)}_\sigma$ (although the weight of the form this is applied to varies) apart from the left factor which is L^1_k. We know L^1_k explicitly from Theorem 4.1. However the choice $p=1$ is not available when this would imply that one of scalars on the left hand side in Theorem 5.2 is zero. In that case, we use the choice $p=2$ which yields the factor $\widetilde{\not\Box}^{(2)}_\sigma$. For all weights concerned we may decompose entirely using, at each step, either $p=1$ or $p=2$.

The scalars on the left hand side in Theorem 5.2 are $k+(\ell-p)-n/2$ and $k-(\ell-p)-n/2$. The latter scalar is always negative as $k \leq n/2$ and $\ell - p \geq 1$. Since we assume $w = k+l-n/2$, the first scalar is equal to $w-p$ so the choice $p=1$ must be avoided only for the weight $w=1$. This affects only the even dimensional case as $w = k+\ell-n/2$. Moreover, for the special case $k=\frac{n}{2}$ we have $w=1$ implies $\ell=1$ and this is known due to Theorem 4.1. Thus, in this process, when $w=1$ (then necessarily n is even and) we are forced to use $p=2$ only if $k \leq n/2-1$.

We shall use the notation

$$P^\Phi_k[E,F] := \prod_{i\in\Phi}\Big[(w-i+1)(w-i+n-2k)E + (w-i)(w-i+n-2k+1)F \tag{29}$$
$$- \frac{2}{n}(w-i)(w-i+1)(w-i+n-2k)(w-i+n-2k+1)J\Big]$$

where $\Phi \subseteq \mathbb{Z}$ is a finite set and $E, F : \mathcal{E}^k[\bar{w}] \to \mathcal{E}^k[\bar{w}]$, $\bar{w} \in \mathbb{R}$ are differential operators. In fact we use this with $E := d\delta$ and $F := \delta d$. Then note that the factor on the right hand side, with fixed $i \in \Phi$, is (up to the multiple $-\frac{1}{k}\sigma$) just $\widetilde{\not\Box}_\sigma$ from (23) where w is replaced by $w-i+1$.

As mentioned above, the only case we need the choice $p=2$ in Theorem 5.2 is on $\mathcal{E}^k[1]$, $k \leq \frac{n}{2}-1$. This leads to the factor $\widetilde{\not\Box}^{(2)}_\sigma$; but this is decomposed in (24). From this it follows that $\widetilde{\not\Box}^{(2)}_\sigma$ is equal (up to a nonzero scalar multiple) to

$$\widetilde{P}_k(E,F) = \Big[(\frac{n}{2}-k-\frac{1}{2})E + (\frac{n}{2}-k+\frac{1}{2})F\Big]\Big[E - F + \frac{4}{n}(\frac{n}{2}-k)J\Big]. \tag{30}$$

where $E = d\delta$ and $F = \delta d$.

With this notation we are ready to state the main factorization result:

Theorem 5.3. *Let $\Phi := \{1,\ldots,\ell\}$ and $w = k+l-n/2$ where $1 \leq k \leq \frac{n}{2}$. The operator*

$$L^\ell_k : \mathcal{E}^k[w] \to \mathcal{E}^k[w-2\ell]$$

has the explicit form

$$L_k^\ell \sim \begin{cases} \sigma^{-\ell}(\widetilde{\not\Box}_\sigma)^\ell \sim (d\delta - \delta d)P_k^{\Phi\setminus\{\ell\}}(d\delta, \delta d) & k = n/2 \\ \sigma^{-\ell}(\widetilde{\not\Box}_\sigma)^\ell \sim P_k^{\Phi}(d\delta, \delta d) & w \leq 0 \wedge k < n/2 \\ \sigma^{-\ell}(\widetilde{\not\Box}_\sigma)^{\ell-w-1}\widetilde{\not\Box}_\sigma^{(2)}(\widetilde{\not\Box}_\sigma)^{w-1} \sim \widetilde{P}_k(d\delta, \delta d)P_k^{\Phi\setminus\{w,w+1\}}(d\delta, \delta d) & w \geq 1 \wedge k < n/2, \end{cases}$$

for n even and

$$L_k^\ell \sim \sigma^{-\ell}(\widetilde{\not\Box}_\sigma)^\ell \sim P_k^{\Phi}(d\delta, \delta d)$$

for n odd. Here $\sim$ means "is equal up to nonzero scalar multiple" and we use the notation from (29) and (30).

Proof. In the case $k = \frac{n}{2}$ we first use Theorem 5.2 with $p = 1$ $(\ell - 1)$-times. This yields $P_k^{\Phi\setminus\{\ell\}}(d\delta, \delta d)$. Then remains, as the leftmost factor, just L_k^1. This is equal to $d\delta - \delta d$ (up to a nonzero scalar) according to Theorem 4.1. The case $w \leq 0$, $k < n/2$ for n even and the case n is odd are even simpler, we simply use Theorem 5.2 ℓ-times with $p = 1$.

Now assume $w \geq 1$ and $k < n/2$. First we apply Theorem 5.2 $(w-1)$ times with $p = 1$; this yields the factor $P_k^{\{1,\ldots,w-1\}}(d\delta, \delta d)$. Then we apply Theorem 5.2 with $p = 2$ which introduces the factor $\widetilde{P}_k(d\delta, \delta d)$. Then we continue with an iterated use of Theorem 5.2 with $p = 1$, and this yields $P_k^{\{w+2,\ldots,\ell\}}(d\delta, \delta d)$. From these three steps, the theorem follows. $\square$

Note the previous theorem provides also factorization of the operators $G_k^{\ell,\sigma}$, due to Theorem 4.2.

Remark 5.4. The explicit formulae for L_k^ℓ show that these operators are formally self-adjoint. Indeed, L_k^ℓ is a polynomial in $E = d\delta$ and $F = \delta d$ where actually only monomials E^p and F^q appear. Thus $F^* = F$ and $E^* = E$ immediately implies that $(L_k^\ell)^* = L_k^\ell$.

6. Decomposition of the null space of L_k^ℓ.

We continue the notation and setting from the previous section. Henceforth shall use notation $\mathcal{N}(F)$ and $\mathcal{R}(F)$ for the null space and range, respectively of an operator F. Recall every operator L_k^ℓ, $1 \leq k \leq \frac{n}{2}$ is a composition of ℓ second order commuting factors, each of them of the form $ad\delta + b\delta d + c$ where a, b, c are (half) integers. In this Section we use our results above to produce a direct sum decomposition of the null space of the operators L_k^ℓ.

6.1. **Riemannian signature and M closed.**

Assuming the Einstein metric has Riemannian signature and that M is closed (i.e. compact, without boundary), we obtain easily an explicit description of $\mathcal{N}(L_k^\ell)$. Recall the space of k-forms decomposes as

$$\mathcal{E}^k = \mathcal{R}(d) \oplus \mathcal{R}(\delta) \oplus (\mathcal{N}(d) \cap \mathcal{N}(\delta))$$

where $\mathcal{N}(d) \cap \mathcal{N}(\delta)$ is the space of harmonic forms. Both $\mathcal{R}(d)$ and $\mathcal{R}(\delta)$ decompose to eigenspaces of the form Laplacian and using the notation

$$\overline{\mathcal{H}}^k_{\sigma,\lambda} := \{f \in \mathcal{E}^k \mid d\delta f = \lambda f\} \subseteq \mathcal{R}(d), \quad \widetilde{\mathcal{H}}^k_{\sigma,\lambda} := \{f \in \mathcal{E}^k \mid \delta d f = \lambda f\} \subseteq \mathcal{R}(\delta)$$
$$\text{and} \quad \mathcal{H}^k_\sigma := \mathcal{N}(d) \cap \mathcal{N}(\delta)$$

from [32], we have

$$\mathcal{N}(ad\delta + b\delta d + c) = \overline{\mathcal{H}}^k_{\sigma,-\frac{c}{a}} \oplus \widetilde{\mathcal{H}}^k_{\sigma,-\frac{c}{b}} \quad \text{and} \quad \mathcal{N}(ad\delta + b\delta d) = \mathcal{H}^k_\sigma \tag{31}$$

for a, b, c nonzero. Using the spectral decomposition given by the form Laplacian with our result Theorem 5.3 we immediately obtain the following.

Theorem 6.1. *Let M be a closed manifold equipped with an Riemannian Einstein metric which is not Ricci flat. Let $w = k + l - n/2$ where $1 \leq k \leq \frac{n}{2}$ and put*

$$\bar{\lambda}_i = \frac{2}{n}(w-i)(w-i+n-2k+1)J \quad \text{and} \quad \tilde{\lambda}_i = \frac{2}{n}(w-i+1)(w-i+n-2k)J,$$

for $1 \leq i \leq \ell$, and $\mu = \frac{4}{n}(\frac{n}{2} - k)J$. Then

$$\mathcal{N}(L^\ell_k) = \begin{cases} \mathcal{H}^k_\sigma \oplus \bigoplus_{i=1}^{\ell-1} \Big(\overline{\mathcal{H}}^k_{\sigma,\bar{\lambda}_i} \oplus \widetilde{\mathcal{H}}^k_{\sigma,\tilde{\lambda}_i} \Big) & k = n/2 \\ \bigoplus_{i=1}^{\ell} \Big(\overline{\mathcal{H}}^k_{\sigma,\bar{\lambda}_i} \oplus \widetilde{\mathcal{H}}^k_{\sigma,\tilde{\lambda}_i} \Big) & w < 0 \wedge k < n/2 \\ \widetilde{\mathcal{H}}^k_{\sigma,0} \oplus \bigoplus_{i=2}^{\ell} \Big(\overline{\mathcal{H}}^k_{\sigma,\bar{\lambda}_i} \oplus \widetilde{\mathcal{H}}^k_{\sigma,\tilde{\lambda}_i} \Big) & w = 0 \\ \mathcal{H}^k_\sigma \oplus \overline{\mathcal{H}}^k_{\sigma,\mu} \oplus \widetilde{\mathcal{H}}^k_{\sigma,-\mu} \oplus \bigoplus_{\substack{i=1,\dots,\ell \\ i \notin \{w,w+1\}}} \Big(\overline{\mathcal{H}}^k_{\sigma,\bar{\lambda}_i} \oplus \widetilde{\mathcal{H}}^k_{\sigma,\tilde{\lambda}_i} \Big) & w \geq 1 \wedge k < n/2, \end{cases}$$

6.2. **The general case.** On general Einstein manifolds we do not have the spectral decomposition but still can reduce the description of the null space of L^ℓ_k to a second order problem. First we show the following:

Proposition 6.2. *Let $L^\ell_k = S_1 \dots S_\ell : \mathcal{E}^k \to \mathcal{E}^k$ be the decomposition of L^ℓ_k to second order factors from Theorem 5.3 in the Einstein metric σ and assume $J \neq 0$ in this scale. Choose a pair of integers $1 \leq t < u \leq \ell$. There are differential operators $\varphi_t, \varphi_u : \mathcal{E}^k \to \mathcal{E}^k$ such that*

$$\varphi_t \circ S_t + \varphi_u \circ S_u = \mathrm{id}$$

where the operators φ_t, φ_u, S_t and S_u mutually commute.

Proof. Considering the right hand side of operators L^ℓ_k in Theorem 5.3, there are several possibilities for second order factors of S_t and S_u. First, they can come from the polynomial P^Φ_k. In this case the right hand side of (29), with $w = k + \ell - \frac{n}{2}$, yields (up to a sign) factors

$$\begin{aligned} \widetilde{S}_i := (A_- + i - 1)(A_+ - i)E + (A_- + i)(A_+ - i + 1)F + \\ + \frac{2}{n}(A_- + i - 1)(A_+ - i)(A_- + i)(A_+ - i + 1)J, \end{aligned} \tag{32}$$

$$\text{where} \quad A_+ = \frac{n}{2} - k + \ell, \quad A_- = \frac{n}{2} - k - \ell, \quad 1 \leq i \leq \ell,\ i \notin \{w, w+1\}.$$

(Note the condition $i \notin \{w, w+1\}$ is vacuous for $w \leq 0$.) We shall call these factors *generic*. Here and below we use the notation $E = d\delta$ and $F = \delta d$, as in the previous Section. Next if not as just described the remaining possible factors are

$$\begin{aligned}
\widetilde{S} &:= (\frac{n}{2} - k - \frac{1}{2})E + (\frac{n}{2} - k + \frac{1}{2})F, \\
\widetilde{S}' &:= E - F + \frac{4}{n}(\frac{n}{2} - k)J \qquad \text{and} \\
\widetilde{S}'' &= E - F.
\end{aligned}$$

Factors $\widetilde{S}$ and $\widetilde{S}'$ come from the polynomial $\widetilde{P}_k$ in Theorem 5.3, i.e. from (30), the factor $\widetilde{S}''$ appears in Theorem 5.3 for $k = \frac{n}{2}$. Summarizing, every factor of S_t and S_u from the Proposition 6.2 is either $\widetilde{S}_i$, $1 \leq i \leq \ell$ (generic factors) or $\widetilde{S}$ or $\widetilde{S}'$ or $\widetilde{S}''$.

First consider a generic pair of factors S_t, S_u i.e. $S_t = \widetilde{S}_i$, $S_u = \widetilde{S}_{i+p}$. That is,

$$\begin{aligned}
S_t =& B_-(B_+ - 1)E + (B_- + 1)B_+F + \frac{2}{n}B_-(B_+ - 1)(B_- + 1)B_+J, \\
S_u =& (B_- + p)(B_+ - p - 1)E + (B_- + p + 1)(B_+ - p)F \\
&+ \frac{2}{n}(B_- + p)(B_+ - p - 1)(B_- + p + 1)(B_+ - p)J.
\end{aligned}$$

Here $B_+ = \frac{n}{2} - k + \ell - r$ and $B_- = \frac{n}{2} - k - \ell + r$ where $r = i - 1$ i.e. $0 \leq r \leq \ell - 1$ and $0 < p$ such that $r + p \leq \ell - 1$. The conditions $i \notin \{w, w+1\}$ and $i + p \notin \{w, w+1\}$ then mean, respectively, $r \notin \{w-1, w\}$ and $r + p \notin \{w-1, w\}$. Now a short computation reveals that

$$\overline{S}_u := \frac{1}{p(B_+ - B_- - (p+1))}\big(S_u - S_t\big) = E + F + \frac{2}{n}\big[2B_+B_- + (p+1)(B_+ - B_- - p)\big]J.$$

where the denominator is nonzero since $B_+ - B_- - (p+1) = 2(\ell - r) - (p+1) = (\ell - r - p - 1) + (\ell - r) > 0$. It is sufficient to replace the pair of operators S_t, S_u by the pair S_t, $\overline{S}_u$ i.e. to find operators φ_t and φ_u such that $\varphi_t \circ S_t + \varphi_u \circ \overline{S}_u = \mathrm{id}$. We shall obtain these operators in the form

$$\varphi_t = x_1E + y_1F + z_1 \quad \text{and} \quad \varphi_u = x_2E + y_2F + z_2 \tag{33}$$

where $x_1, y_1, z_1, x_2, y_2, z_2 \in \mathbb{R}$. We will use the notation

$$\begin{aligned}
&\alpha_1 = B_-(B_+ - 1) \neq 0, \quad \alpha_2 = B_+(B_- + 1) \neq 0 \qquad \text{and} \\
&\beta = \frac{2}{n}\big[2B_+B_- + (p+1)(B_+ - B_- - p)\big].
\end{aligned}$$

The inequalities follow from the following: $B_+ > 0$ always and $B_+ = (\frac{n}{2} - k) + (\ell - r) = 1$ only for $k = \frac{n}{2}$ and $\ell = r + 1$ and hence $w = k + \ell - \frac{n}{2} = r + 1$. Next $B_- = 0$ is equivalent to $r = k + \ell - \frac{n}{2} = w$ and $B_- = -1$ is equivalent to $r = k + \ell - \frac{n}{2} - 1 = w - 1$. (Recall we assume $r \notin \{w-1, w\}$.)

It remains to determine the scalars $x_1, y_1, z_1, x_2, y_2, z_2$. Putting $x_2 = -\alpha_1 x_1$, $y_2 = -\alpha_2 y_1$ and $z_2 = 0$ we obtain

$$\begin{aligned}
&\varphi_t \circ S_t + \varphi_u \circ \overline{S}_u = \\
&= (x_1E + y_1F + z_1) \circ (\alpha_1 E + \alpha_2 F + \frac{2}{n}\alpha_1\alpha_2 J) + (-\alpha_1 x_1 E - \alpha_2 y_1 F) \circ (E + F + \beta J) = \\
&= \big[\big(\frac{2}{n}\alpha_1\alpha_2 - \beta\alpha_1\big) J x_1 + \alpha_1 z_1\big] E + \big[\big(\frac{2}{n}\alpha_1\alpha_2 - \beta\alpha_2\big) J y_1 + \alpha_2 z_1\big] F + \frac{2}{n}\alpha_1\alpha_2 z_1 J \,\mathrm{id}\,.
\end{aligned}$$

We are looking for x_1, y_1, z_1 and z_2 such that only the last term of the previous display is nonzero (and equal to the identity). Thus $z_1 := \frac{n}{2\alpha_1\alpha_2 J}$ and this determines x_1 and y_1 provided

$$\alpha_1\big[\frac{2}{n}\alpha_2 - \beta\big] \neq 0 \quad \text{and} \quad \alpha_2\big[\frac{2}{n}\alpha_1 - \beta\big] \neq 0.$$

We shall now verify the previous two inequalities. First we consider

$$\alpha_1\big[\frac{2}{n}\alpha_2 - \beta\big] = -\frac{2}{n}\alpha_1\big[B_+B_- + (p+1)(B_+ - B_- - p) - B_+\big]$$

where $\alpha_1 \neq 0$. Observe that the second factor on the right hand side is equal to zero only for $B_- = -p$ or $B_+ = p+1$. (Considering the second factor as the quadratic expression in p, these are the two roots.) The case $B_+ - p - 1 = (\frac{n}{2} - k) + (\ell - r - p - 1) = 0$ is equivalent to $\frac{n}{2} = k$ and $\ell = r + p + 1$ i.e. $w = k + \ell - \frac{n}{2} = r + p + 1$ and the case $B_- = -p$ is equivalent to $-w = \frac{n}{2} - k - \ell = -r - p$. Since we assume $r + p \notin \{w-1, w\}$, the previous display is indeed nonzero. Second we consider

$$\alpha_2\big[\frac{2}{n}\alpha_1 - \beta\big] = -\frac{2}{n}\alpha_2\big[B_+B_- + (p+1)(B_+ - B_- - p) + B_-\big]$$

where $\alpha_2 \neq 0$. Observe now that the second factor on the right hand side is equal to zero only for $B_- = -p-1$ or $B_+ = p$. Since $B_+ - p = (\frac{n}{2} - k) + (\ell - r - p) > 0$ and $B_- = -p$ is equivalent to $-w + 1 = \frac{n}{2} - k - \ell + 1 = -r - p$ (and we assume $r + p \notin \{w-1, w\}$), the previous display is also nonzero. Summarizing, we have proved the proposition for generic factors S_t, S_u. In particular, we have proved the proposition for n odd.

Henceforth we assume n even. The remaining possibilities for how to chose the pair of operators S_t, S_u are:

$$\begin{aligned}
&(a)\ S_t \text{ generic and } S_u = \widetilde{S} \text{ for } k < \frac{n}{2}, \\
&(b)\ S_t \text{ generic and } S_u = \widetilde{S}' \text{ for } k < \frac{n}{2}, \\
&(c)\ S_t = \widetilde{S} \text{ and } S_u = \widetilde{S}' \text{ for } k < \frac{n}{2}, \\
&(d)\ S_t \text{ generic and } S_u = \widetilde{S}'' \text{ for } k = \frac{n}{2}.
\end{aligned}$$

We shall discuss cases (a), (b), (c) first and assume $k < \frac{n}{2}$. Then we will treat (d) separately. In the case (a) we deal with $S_t = \alpha_1 E + \alpha_2 F + \frac{2}{n}\alpha_1\alpha_2 J$ and $S_u = aE + bF$ where $a = \frac{n}{2} - k - \frac{1}{2} \neq 0$ and $b = \frac{n}{2} - k + \frac{1}{2} \neq 0$ (since n is even).

We find operators φ_u and φ_t in the form (33) where $x_2 = -\alpha_1\frac{x_1}{a}$, $y_2 = -\alpha_1\frac{y_1}{b}$ and $z_2 = 0$. Then

$$
\begin{aligned}
&\varphi_t \circ S_t + \varphi_u \circ S_u = \\
&\quad = (x_1E + y_1F + z_1) \circ (\alpha_1E + \alpha_2F + \frac{2}{n}\alpha_1\alpha_2J) + (-\alpha_1\frac{x_1}{a}E - \alpha_1\frac{y_1}{b}F) \circ (aE + bF) = \\
&\quad = \big[\frac{2}{n}\alpha_1\alpha_2Jx_1 + \alpha_1z_1\big]E + \big[\frac{2}{n}\alpha_1\alpha_2Jy_1 + \alpha_2z_1\big]F + \frac{2}{n}\alpha_1\alpha_2z_1J\,\mathrm{id}\,.
\end{aligned}
$$

We need the previous display to be equal to identity and since $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$, this can be obviously achieved for unique z_1, x_1 and y_1.

In the case (b) we have $S_t = \alpha_1E+\alpha_2F+\frac{2}{n}\alpha_1\alpha_2J$ and $S_u = E-F+\frac{4}{n}(\frac{n}{2}-k)J$. In this case we find operators φ_u and φ_t in the form (33) where $x_2 = -\alpha_1x_1$, $y_2 = \alpha_2y_1$ and $z_2 = 0$. Then putting $\gamma = 2(\frac{n}{2} - k) \neq 0$ we obtain

$$
\begin{aligned}
&\varphi_t \circ S_t + \varphi_u \circ S_u = \\
&\quad = (x_1E + y_1F + z_1) \circ (\alpha_1E + \alpha_2F + \frac{2}{n}\alpha_1\alpha_2J) + (-\alpha_1x_1E + \alpha_2y_1F) \circ (E - F + \frac{2}{n}\gamma J) = \\
&\quad = \big[\frac{2}{n}\alpha_1\big(\alpha_2 - \gamma\big)Jx_1 + \alpha_1z_1\big]E + \big[\frac{2}{n}\alpha_2\big(\alpha_1 + \gamma\big)Jy_1 + \alpha_1z_1\big]F + \frac{2}{n}\alpha_1\alpha_2z_1J\,\mathrm{id}\,.
\end{aligned}
$$

We need the previous display to be equal to identity and this (uniquely) determines z_1, x_1 and y_1 provided $\alpha_2 - \gamma \neq 0$ and $\alpha_1 + \gamma \neq 0$ where, observe, $\gamma = B_+ + B_-$. Thus $\alpha_2 - \gamma = B_+(B_- + 1) - (B_+ + B_-) = B_-(B_+ - 1)$ and $\alpha_1 + \gamma = B_-(B_+ - 1) + (B_+ + B_-) = B_+(B_- + 1)$. Since $B_- \neq 0$, $B_+ \neq 0$ and we observed above that both $B_+ = 1$ $B_- = -1$ imply $r = w - 1$ (recall we assume $r \notin \{w - 1, w\}$), we have indeed verified that $\alpha_2 - \gamma \neq 0$ and $\alpha_1 + \gamma \neq 0$.

In the case (c) we have $S_t = aE + bF$ and $S_u = E - F + \frac{4}{n}(\frac{n}{2} - k)J$ where $a = \frac{n}{2} - k - \frac{1}{2} \neq 0$ and $b = \frac{n}{2} - k + \frac{1}{2} \neq 0$. Here we find operators φ_u and φ_t in the form (33) where $x_2 = -ax_1$, $y_2 = by_1$ and $z_1 = 0$. Then putting $\gamma = 2(\frac{n}{2} - k) \neq 0$ we obtain

$$
\begin{aligned}
&\varphi_t \circ S_t + \varphi_u \circ S_u = \\
&\quad = (x_1E + y_1F) \circ (aE + bF) + (-ax_1E + by_1F + z_2) \circ (E - F + \frac{2}{n}\gamma J) = \\
&\quad = \big[-\frac{2}{n}a\gamma Jx_1 + z_2\big]E + \big[\frac{2}{n}b\gamma Jy_1 - z_2\big]F + \frac{2}{n}\gamma z_2J\,\mathrm{id}\,.
\end{aligned}
$$

One can obviously choose (uniquely) x_1, y_1 and z_2 such that the previous display is equal to the identity.

Finally we shall discuss the case (d). We thus assume $k = \frac{n}{2}$. Then $B_+ = \ell - r$ and $B_- = -(\ell - r)$ hence $\alpha_1 = \alpha_2 = -(\ell - r)(\ell - r - 1) =: \bar{\alpha} \neq 0$. Here the inequality follows since $\ell - r > 0$ and $\ell - r - 1 = 0$ would imply $w = k + \ell - \frac{n}{2} = r + 1$ (recall we assume $r \notin \{w - 1, w\}$). Thus we can replace $S_t = \bar{\alpha}^2(E + F + \frac{2}{n}J)$ by $\overline{S}_t := E + F + \frac{2}{n}J$. The other factor is $S_u = E - F$. We find operators φ_u and φ_t

in the form (33) where $x_2 = x_1$, $y_2 = y_1$ and $z_2 = 0$. Then

$$\begin{aligned}&\varphi_t \circ \overline{S}_t + \varphi_u \circ S_u = \\ &\quad = (x_1 E + y_1 F + z_1) \circ (E + F + \frac{2}{n} J) + (-x_1 E + y_1 F) \circ (E - F) = \\ &\quad = \big[\frac{2}{n} J x_1 + z_1\big] E + \big[\frac{2}{n} J y_1 + z_1\big] F + \frac{2}{n} J z_1 \,\mathrm{id}\,.\end{aligned}$$

It is obvious that there is a unique choice for x_1, y_1 and z_1 such that the previous display is equal to the identity. □

Combining the previous proposition with [31, Corollary 2.3], we obtain the final result.

Theorem 6.3. *Let $L_k^\ell = S_1 \dots S_\ell : \mathcal{E}^k \to \mathcal{E}^k$ be the decomposition of L_k^ℓ to second order factors from Theorem 5.3 in the Einstein scale σ and assume $J \neq 0$ in this scale. Then*

$$\mathcal{N}(L_k^\ell) = \mathcal{N}(S_1) \oplus \dots \oplus \mathcal{N}(S_\ell).$$

7. The Fefferman-Graham ambient metric

Thus let us review briefly the basic relationship between the Fefferman-Graham ambient metric construction and tractor calculus as described in [15, 28] for general conformal manifolds.

Let $\pi : \mathcal{Q} \to M$ be a conformal structure of signature (p, q). Let us use ρ to denote the $\mathbb{R}_+$ action on $\mathcal{Q}$ given by $\rho(s)(x, g_x) = (x, s^2 g_x)$. An *ambient manifold* is a smooth $(n+2)$-manifold $\tilde{M}$ endowed with a free $\mathbb{R}_+$–action ρ and an $\mathbb{R}_+$–equivariant embedding $i : \mathcal{Q} \to \tilde{M}$. We write $\boldsymbol{X} \in \mathfrak{X}(\tilde{M})$ for the fundamental field generating the $\mathbb{R}_+$–action, that is for $f \in C^\infty(\tilde{M})$ and $u \in \tilde{M}$ we have $\boldsymbol{X} f(u) = (d/dt) f(\rho(e^t)u)|_{t=0}$.

If $i : \mathcal{Q} \to \tilde{M}$ is an ambient manifold, then an *ambient metric* is a pseudo–Riemannian metric $\boldsymbol{h}$ of signature $(p+1, q+1)$ on $\tilde{M}$ such that the following conditions hold:

(i) The metric $\boldsymbol{h}$ is homogeneous of degree 2 with respect to the $\mathbb{R}_+$–action, i.e. if $\mathcal{L}_{\boldsymbol{X}}$ denotes the Lie derivative by $\boldsymbol{X}$, then we have $\mathcal{L}_{\boldsymbol{X}} \boldsymbol{h} = 2\boldsymbol{h}$. (I.e. $\boldsymbol{X}$ is a homothetic vector field for h.)
(ii) For $u = (x, g_x) \in \mathcal{Q}$ and $\xi, \eta \in T_u \mathcal{Q}$, we have $\boldsymbol{h}(i_* \xi, i_* \eta) = g_x(\pi_* \xi, \pi_* \eta)$.
To simplify the notation we will usually identify $\mathcal{Q}$ with its image in $\tilde{M}$ and suppress the embedding map i.

To link the geometry of the ambient manifold to the underlying conformal structure on M one requires further conditions. In [19] Fefferman and Graham treat the problem of constructing a formal power series solution along $\mathcal{Q}$ for the Goursat problem of finding an ambient metric $\boldsymbol{h}$ satisfying (i) and (ii) and the condition that it be Ricci flat, i.e. $\mathrm{Ric}(\boldsymbol{h}) = 0$. In even dimensions for a general conformal structure this is obstructed at finite order.

However when the underlying conformal structure is (conformally) Einstein then an explicit Ricci-flat ambient metric is available [36, 40, 41]. Here we shall use only

the existence part of Ricci-flat ambient metric. The uniqueness of the operators we will construct is a consequence of the fact that they can be uniquely expressed in terms of the underlying conformal structure as in [15, 28].

It turns out that in metrics satisfying these conditions $Q := \boldsymbol{h}(\boldsymbol{X}, \boldsymbol{X})$ is a defining function for $\mathcal{Q}$ and $2\boldsymbol{h}(\boldsymbol{X}, \cdot) = dQ$ to all orders in odd dimensions and up to the addition of terms vanishing to order $n/2$ in even dimensions. We write $\boldsymbol{\nabla}$ for the ambient Levi-Civita connection determined by $\boldsymbol{h}$. We will use and use upper case abstract indices $A, B, \cdots$ for tensors on $\tilde{M}$. For example, if v^B is a vector field on $\tilde{M}$, then the ambient Riemann tensor will be denoted $\boldsymbol{R}_{AB}{}^{C}{}_{D}$ and defined by $[\boldsymbol{\nabla}_A, \boldsymbol{\nabla}_B]v^C = \boldsymbol{R}_{AB}{}^{C}{}_{D}v^D$. In this notation the ambient metric is denoted $\boldsymbol{h}_{AB}$ and, with its inverse, this is used to raise and lower indices in the usual way. We will not normally distinguish tensors related in this way even in index free notation; the meaning should be clear from the context. Thus for example we shall use $\boldsymbol{X}$ to mean both the Euler vector field $\boldsymbol{X}^A$ and the 1-form $\boldsymbol{X}_A = \boldsymbol{h}_{AB}\boldsymbol{X}^B$.

Let $\tilde{\mathcal{E}}(w)$ denote the space of functions on $\tilde{M}$ which are homogeneous of degree $w \in \mathbb{R}$ with respect to the action ρ. That is $f \in \tilde{\mathcal{E}}(w)$ means that $\boldsymbol{X}f = wf$. Similarly a tensor field F on $\tilde{M}$ is said to be *homogeneous of degree* w if $\rho(s)^*F = s^wF$ or equivalently $\mathcal{L}_{\boldsymbol{X}}F = wF$. Just as sections of $\mathcal{E}[w]$ are equivalent to functions in $\tilde{\mathcal{E}}(w)|_{\mathcal{Q}}$ we will see that the restriction of homogeneous tensor fields to $\mathcal{Q}$ have interpretations on M as weighted sections of a tractor bundles [15, 28].

On the ambient tangent bundle $T\tilde{M}$ we define an action of $\mathbb{R}_+$ by $s \cdot \xi := s^{-1}\rho(s)_*\xi$. The sections of $T\tilde{M}$ which are fixed by this action are those which are homogeneous of degree -1. Let us denote by $\boldsymbol{\mathcal{T}}$ the space of such sections and write $\boldsymbol{\mathcal{T}}(w)$ for sections in $\boldsymbol{\mathcal{T}} \otimes \tilde{\mathcal{E}}(w)$, where the $\otimes$ here indicates a tensor product over $\tilde{\mathcal{E}}(0)$. Along $\mathcal{Q}$ the $\mathbb{R}_+$ action on $T\tilde{M}$ is compatible with the $\mathbb{R}_+$ action on $\mathcal{Q}$, so the quotient $(T\tilde{M}|_{\mathcal{Q}})/\mathbb{R}_+$, is a rank $n+2$ vector bundle over $\mathcal{Q}/\mathbb{R}_+ = M$; in fact this is (up to isomorphism) the normal standard tractor bundle $\mathcal{T}$ (or $\mathcal{E}^A$) [15, 28] and the composition structure of $\mathcal{T}$ reflects the vertical subbundle $T\mathcal{Q}$ in $T\tilde{M}|_{\mathcal{Q}}$. Sections of $\mathcal{T}$ are equivalent to sections of $T\tilde{M}|_{\mathcal{Q}}$ which are homogeneous of degree -1, that is sections of $\boldsymbol{\mathcal{T}}$. Using this relationship one sees that the ambient metric $\boldsymbol{h}$ and the ambient connection $\boldsymbol{\nabla}$ descend to, respectively the tractor metric h, and the tractor connection $\nabla^{\mathcal{T}}$. For the metric this is obvious. We discuss the connection briefly. For $U \in \mathcal{T}$, let $\tilde{U}$ be the corresponding section of $\boldsymbol{\mathcal{T}}$. A tangent vector field ξ on M has a lift to a homogeneous degree 0 section $\tilde{\xi}$, of $T\tilde{M}|_{\mathcal{Q}}$, which is everywhere tangent to $\mathcal{Q}$. This is unique up to adding $f\boldsymbol{X}$, where $f \in \tilde{\mathcal{E}}(0)|_{\mathcal{Q}}$. We extend $\tilde{U}$ and $\tilde{\xi}$ smoothly and homogeneously to fields on $\tilde{M}$. Then we can form $\boldsymbol{\nabla}_{\tilde{\xi}}\tilde{U}$; along $\mathcal{Q}$, this is clearly independent of the extensions. Since $\boldsymbol{\nabla}_{\boldsymbol{X}}\tilde{U} = 0$, the section $\boldsymbol{\nabla}_{\tilde{\xi}}\tilde{U}$ is also independent of the choice of $\tilde{\xi}$ as a lift of ξ. Finally, the restriction of $\boldsymbol{\nabla}_{\tilde{\xi}}\tilde{U}$ is a homogeneous degree -1 section of $T\tilde{M}|_{\mathcal{Q}}$ and so determines a section of $\mathcal{T}$ which depends only on U and ξ. This is $\nabla^{\mathcal{T}}U$.

Finally we will say that an ambient tensor F is homogeneous of *weight* w if $\boldsymbol{\nabla}_{\boldsymbol{X}}F = wF$. The weight is a convenient shifting of homogeneity degree. Note,

for example, that an ambient 1-form $\tilde{U}$ which is homogeneous of degree -1 is homogeneous of weight 0 and this means that $\boldsymbol{\nabla}_{\boldsymbol{X}}\tilde{U} = 0$.

7.1. **Factorisation.** Here we prove by induction that the conformal operators on forms factorise. The point is that although the operators on tractors factor we need see how this translates into a factorisation of the operators on forms on the base manifold.

On the ambient manifold a special role is played by differential operators P on ambient tensor bundles which act *tangentially* along $\mathcal{Q}$, in the sense that $PQ = QP'$ for some operator P' (or equivalently $[P, Q] = QP''$ for some P''). Note that compositions of tangential operators are tangential. If tangential operators are homogeneous (i.e. the commutator with the Lie derivative $\boldsymbol{\mathcal{L}}_{\boldsymbol{X}}$ recovers a constant multiple of the operator) then they descend to operators on on M. An example of a tangential operator is given by

$$(n + 2w - 2)\boldsymbol{\nabla} + \boldsymbol{X}\not{\boldsymbol{\Delta}} =: \not{\boldsymbol{D}} \colon \boldsymbol{\mathcal{T}}^{\Phi}(w) \to \boldsymbol{\mathcal{T}} \otimes \boldsymbol{\mathcal{T}}^{\Phi}(w - 1)$$

where $\boldsymbol{\mathcal{T}}^{\Phi}(w)$ indicates the space of sections, homogeneous of weight w, of some ambient tensor bundle, and

$$\not{\boldsymbol{\Delta}} = \boldsymbol{\Delta} - \boldsymbol{R}\sharp\sharp.$$

We will leave the verification that $\not{\boldsymbol{D}}$ is tangential to the reader, but note that is also follows from the result that $(n+2w-2)\boldsymbol{\nabla} + \boldsymbol{X}\boldsymbol{\Delta} =: \boldsymbol{D} : \boldsymbol{\mathcal{T}}^{\Phi}(w) \to \boldsymbol{\mathcal{T}} \otimes \boldsymbol{\mathcal{T}}^{\Phi}(w-1)$ is tangential as discussed in [28, 15]. Since this is tangential and homogeneous it descends to an operator on weighted tractors. In fact it gives the usual tractor-D [28, 15]. The ambient $\boldsymbol{R}\sharp\sharp$ similarly descends (in dimensions $n \neq 4$) to a multiple of $W\sharp\sharp$. Thus acting on weighted tractor bundles [29]. Thus $\boldsymbol{\mathcal{T}}^{\Phi}(w) \to \boldsymbol{\mathcal{T}} \otimes \boldsymbol{\mathcal{T}}^{\Phi}(w-1)$ descends to $\mathcal{T}^{\Phi}(w) \to \mathcal{T} \otimes \mathcal{T}^{\Phi}(w-1)$ in dimensions other than 4. (Here $\mathcal{T}^{\Phi}$ means the tractor bundle corresponding to $\boldsymbol{\mathcal{T}}^{\Phi}$.) Henceforth for $(M, [g])$ of dimension 4 we take $\not{\boldsymbol{D}} := \boldsymbol{D}$, rather then the definition above.

Now if $(M, [g])$ is conformally Einstein and I a parallel tractor corresponding to an Einstein scale then along $\mathcal{Q}$ in $\tilde{M}$ we have a corresponding parallel vector field $\boldsymbol{I}$. From the explicit formula for the ambient metric on over an Einstein manifold ones sees that $\boldsymbol{I}$ extends to a parallel vector field on $\tilde{M}$. (In fact when the Einstein scale is not Ricci flat then the ambient metric is given as a product of the metric cone with a line.) We have (on $\boldsymbol{\mathcal{T}}^{\Phi}[w]$)

$$\boldsymbol{I}^A \not{\boldsymbol{D}}_A = (n + 2w - 2)\boldsymbol{I}^A \boldsymbol{\nabla}_A + \boldsymbol{\sigma}\not{\boldsymbol{\Delta}}\,,$$

where $\boldsymbol{\sigma} = \boldsymbol{I}_A \boldsymbol{X}^A \in \tilde{\mathcal{E}}(1)$. Note that $\boldsymbol{\sigma}$ is a homogeneous function on $\mathcal{Q}$ corresponding to $\sigma = I_A X^A$.

Thus if we extend a tensor field $\boldsymbol{U} \in \boldsymbol{\mathcal{T}}^{\Phi}(w)|_{\mathcal{Q}}$ off $\mathcal{Q}$ in such a way that $\boldsymbol{I}^A \boldsymbol{\nabla}_A \boldsymbol{U} = 0$ (which implies $\boldsymbol{U} \in \boldsymbol{\mathcal{T}}^{\Phi}(w)$) then we get simply

$$\boldsymbol{I}^A \not{\boldsymbol{D}}_A = \boldsymbol{\sigma}\not{\boldsymbol{\Delta}}\,.$$

Note that $\boldsymbol{I}^A \boldsymbol{\nabla}_A \boldsymbol{U} = 0$ can be achieved by starting with a section along $\mathcal{Q}$ and then extending off $\mathcal{Q}$ by parallel transport. The key point here is that $I^A X_A$ is non-vanishing, at least in a neighbourhood of $\mathcal{Q}$, and so $\boldsymbol{I}^A \boldsymbol{\nabla}_A$ is not tangential to $\mathcal{Q}$.

Next observe that, since $\boldsymbol{\sigma} = \boldsymbol{I}_A \boldsymbol{X}^A$ and $\boldsymbol{I}_A$ is parallel, we have

$$\boldsymbol{\nabla}_A \boldsymbol{\sigma} = \boldsymbol{I}_A \ ,$$

which is parallel. Thus

$$[\not{\boldsymbol{\Delta}}, \boldsymbol{\sigma}] = [\boldsymbol{\Delta}, \boldsymbol{\sigma}] = 2\boldsymbol{I}^A \boldsymbol{\nabla}_A \tag{34}$$

where we consider $\boldsymbol{\sigma}$ as a multiplication operator.

The following observations will be useful.

Lemma 7.1. *If $\boldsymbol{R}$ denotes the ambient curvature then $\boldsymbol{I}^A \boldsymbol{\nabla}_A \boldsymbol{R} = 0$.*

Proof. By the Bianchi identity

$$\boldsymbol{I}^A \boldsymbol{\nabla}_A \boldsymbol{R}_{BC}{}^D{}_E + \boldsymbol{I}^A \boldsymbol{\nabla}_C \boldsymbol{R}_{AB}{}^D{}_E + \boldsymbol{I}^A \boldsymbol{\nabla}_B \boldsymbol{R}_{CA}{}^D{}_E = 0.$$

But $\boldsymbol{I}$ is parallel which implies that $[\boldsymbol{\nabla}, \boldsymbol{I}] = 0$ and $\boldsymbol{I}^A \boldsymbol{R}_{AB}{}^D{}_E = 0 = \boldsymbol{I}^A \boldsymbol{R}_{CA}{}^D{}_E$. So the result follows. □

Lemma 7.2. *If $\boldsymbol{U}$ is an ambient tensor such that $\boldsymbol{I}^A \boldsymbol{\nabla}_A \boldsymbol{U} = 0$ then, for any $p \in \mathbb{N} \cup \{0\}$, $\boldsymbol{I}^A \boldsymbol{\nabla}_A (\not{\boldsymbol{\Delta}}^p \boldsymbol{U}) = 0$*

Proof. Clearly acting on any ambient tensor we have $[\boldsymbol{I}^A \boldsymbol{\nabla}_A, \boldsymbol{\nabla}_B] = 0$. Thus $\boldsymbol{I}^A \boldsymbol{\nabla}_A$ commutes with the Bochner Laplacian $\boldsymbol{\Delta}$. On the other hand by definition $\not{\boldsymbol{\Delta}}$ differs from the Bochner by a curvature action: $\not{\boldsymbol{\Delta}} - \boldsymbol{\Delta} = -\boldsymbol{R}\sharp\sharp$, while from the previous Lemma the ambient curvature is parallel along the flow of $\boldsymbol{I}^A \boldsymbol{\nabla}_A$. □

The main technical result we need is this.

Proposition 7.3. *For $\boldsymbol{f}$ an ambient form homogeneous of weight $k - n/2$ we have*

$$(\boldsymbol{I}^A \not{\boldsymbol{D}}_A)^k \boldsymbol{f} = \boldsymbol{\sigma}^k \not{\boldsymbol{\Delta}}^k \boldsymbol{f} \ ,$$

along $\mathcal{Q}$.

Proof. First note that both sides are tangential operators. For the right-hand-side this is in [6]. For the left-hand-side it holds simply because $\not{\boldsymbol{D}}$ is tangential and $\boldsymbol{I}$ is parallel on the ambient manifold. So neither side can depend on the transverse (to $\mathcal{Q}$) derivatives of the homogeneous $\boldsymbol{f}$.

Now the result is true if $k = 1$. Also, calculating along $\mathcal{Q}$,

$$(\boldsymbol{I}^A \not{\boldsymbol{D}}_A)^k \boldsymbol{f} = (\boldsymbol{I}^B \not{\boldsymbol{D}}_B)^{k-1} \boldsymbol{I}^A \not{\boldsymbol{D}}_A \boldsymbol{f}$$

and so by induction

$$(\boldsymbol{I}^A \not{\boldsymbol{D}}_A)^k \boldsymbol{f} = \boldsymbol{\sigma}^{k-1} \not{\boldsymbol{\Delta}}^{k-1} \boldsymbol{I}^A \not{\boldsymbol{D}}_A \boldsymbol{f} \ .$$

Since the result is independent of transverse derivatives we may choose the extension off $\mathcal{Q}$ to suit. Thus we assume without loss of generality that $\boldsymbol{I}^A \boldsymbol{\nabla}_A \boldsymbol{f} = 0$. Then $\boldsymbol{I}^A \not{\boldsymbol{D}}_A \boldsymbol{f} = \boldsymbol{\sigma} \not{\boldsymbol{\Delta}} \boldsymbol{f}$ and so

$$\boldsymbol{\sigma}^{k-1} \not{\boldsymbol{\Delta}}^{k-1} (\boldsymbol{I}^A \not{\boldsymbol{D}}_A) \boldsymbol{f} = \boldsymbol{\sigma}^{k-1} \not{\boldsymbol{\Delta}}^{k-1} (\boldsymbol{\sigma} \not{\boldsymbol{\Delta}} \boldsymbol{f}).$$

So from (34) and Lemma 7.2 the result follows. □

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